Integer Programming Games*

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Dagstuhl Seminar #22441 - Optimization at the Second Level

CANADA EXCELLENCE RESEARCH CHAIR
DATA SCIENCE FOR REAL-TIME DECISION-MAKING

PRINCETON UNIVERSITY

Integer Programming Games*

*And many speculations about ***Robust Optimization, Bilevel Programming, Nash equilibria and Generalized Nash equilibria.***

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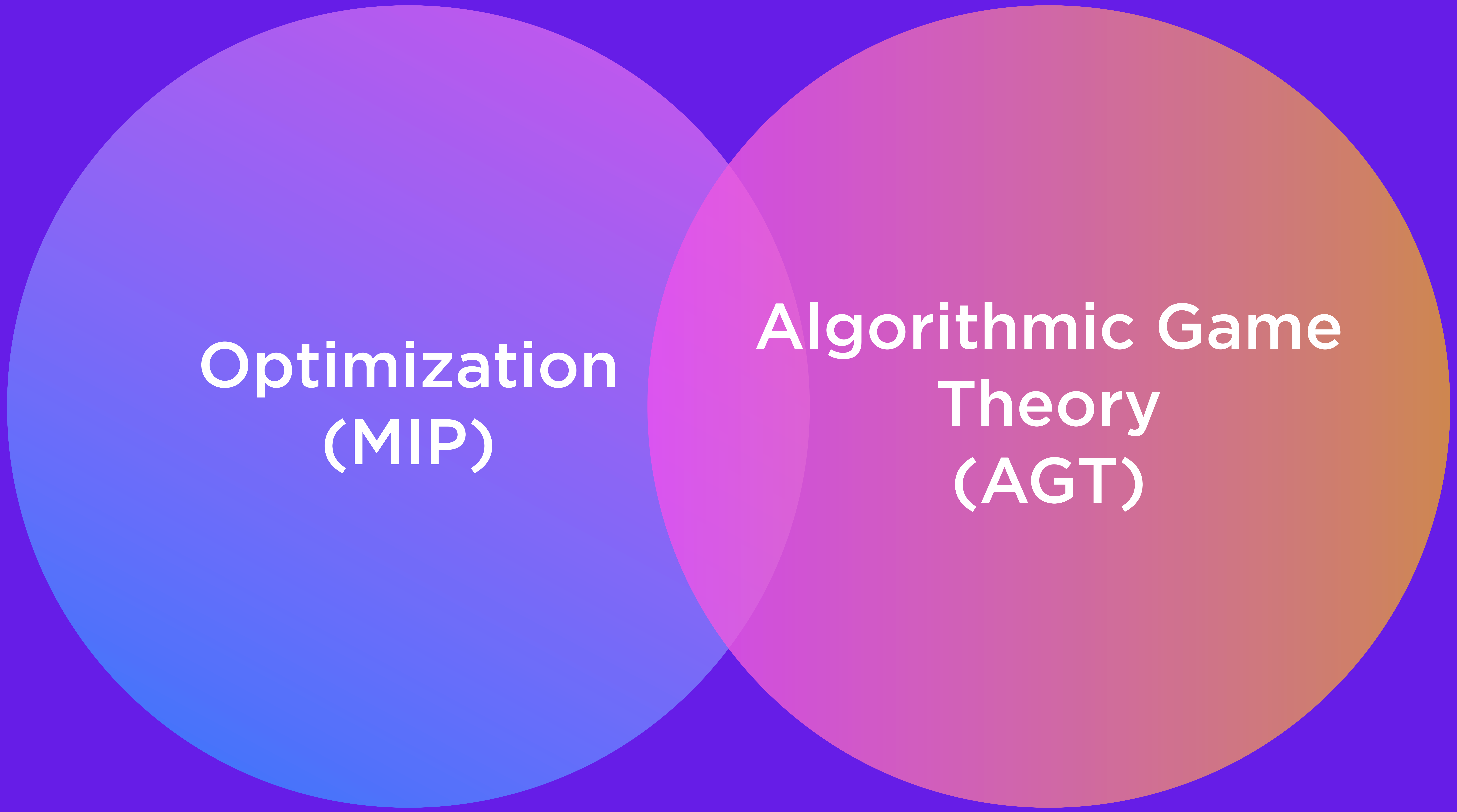

Optimization
(MIP)
Algorithmic Game
Theory
(AGT)

In this talk

I'll try to convince you that Integer Programming Games are:

- Mathematically and conceptually connected with ***Robust and Bilevel Optimization***
- Another way to frame ***"structured" uncertainty***
- A natural ***multi-agent extension*** of Combinatorial Optimization
- ***At the second level***
- A Cool 😎 area of research we should get explore!

I'll also try to use more images and less math since it's Thursday evening…

Unless specified, the (*most of the*) games of this talk are *simultaneous*

As standard game theory/bilevel notation, let x^i denote the vector of variables of player i, and let the operator $(\cdot)^{-i}$ be $(\cdot)$ except i

Decision-making is rarely an individual task.

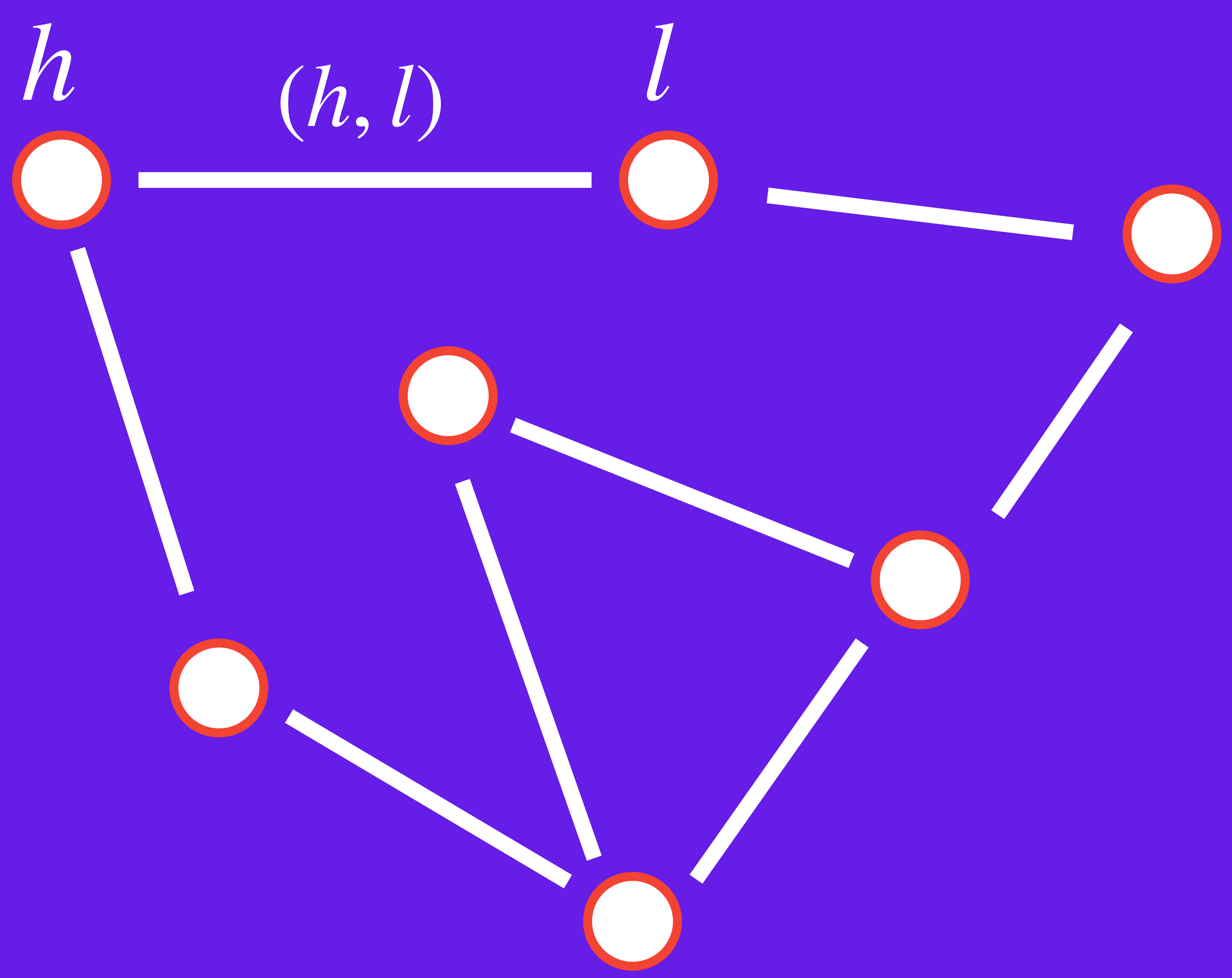
Uncertainty

Interactions with other decision-makers

Time-evolving dynamics

Network Formation

Network Formation Game



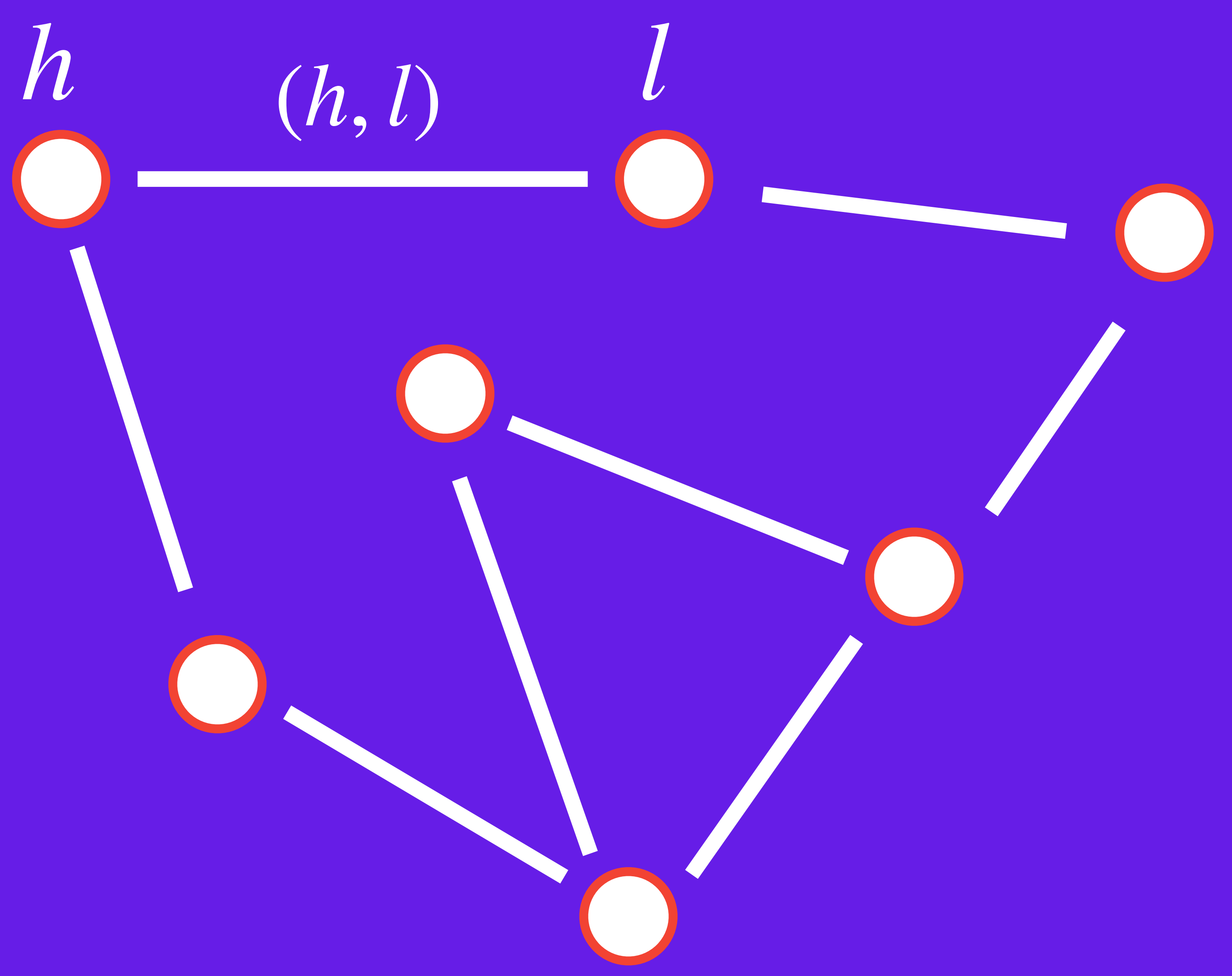
There are n players optimizing simultaneously the shortest path on a graph $G = (V, E)$ so that:

- Any $(h, l) \in E : h, l \in V$ has a cost $c_{hl} \in \mathbb{Z}^+$
- The player i needs to go **from** s^i **to** t^i
- Player i **has a weight** w^i

The cost of each edge is **split proportionally to each player's weight**

Chen and Roughgarden (2006), Anshelevich et al. (2008), Nisan et al. (2008)

Network Formation Game



$$\min_{x^i}\Big\{\sum_{(h,l)\in E}\frac{w^i c_{hl} x^i_{hl}}{\sum_{k=1}^n w^k x^k_{hl}} : x^i \in \mathcal{X}^i\Big\}.$$

$x^i_{hl} = 1$ iff player i selects edge $(h, l) \in E$

$\mathcal{X}^i$ are linear flow constraints for the path $s^i \to t^i$

Chen and Roughgarden (2006), Anshelevich et al. (2008), Nisan et al. (2008)

Multi-agent Assortment

$$\max_{x^1} \quad 6x_1^1 + x_2^1$$

$$\text{s.t.} \quad 3x_1^1 + 2x_2^1 \le 4$$

$$x^1 \in \{0,1\}^2$$

Their profits **interact**

$$\max_{x^1} \quad 6x_1^1 + x_2^1 - 4x_1^1 x_1^2 + 6x_2^1 x_2^2$$

$$\text{s.t.} \quad 3x_1^1 + 2x_2^1 \leq 4$$

$$x^1 \in \{0,1\}^2$$

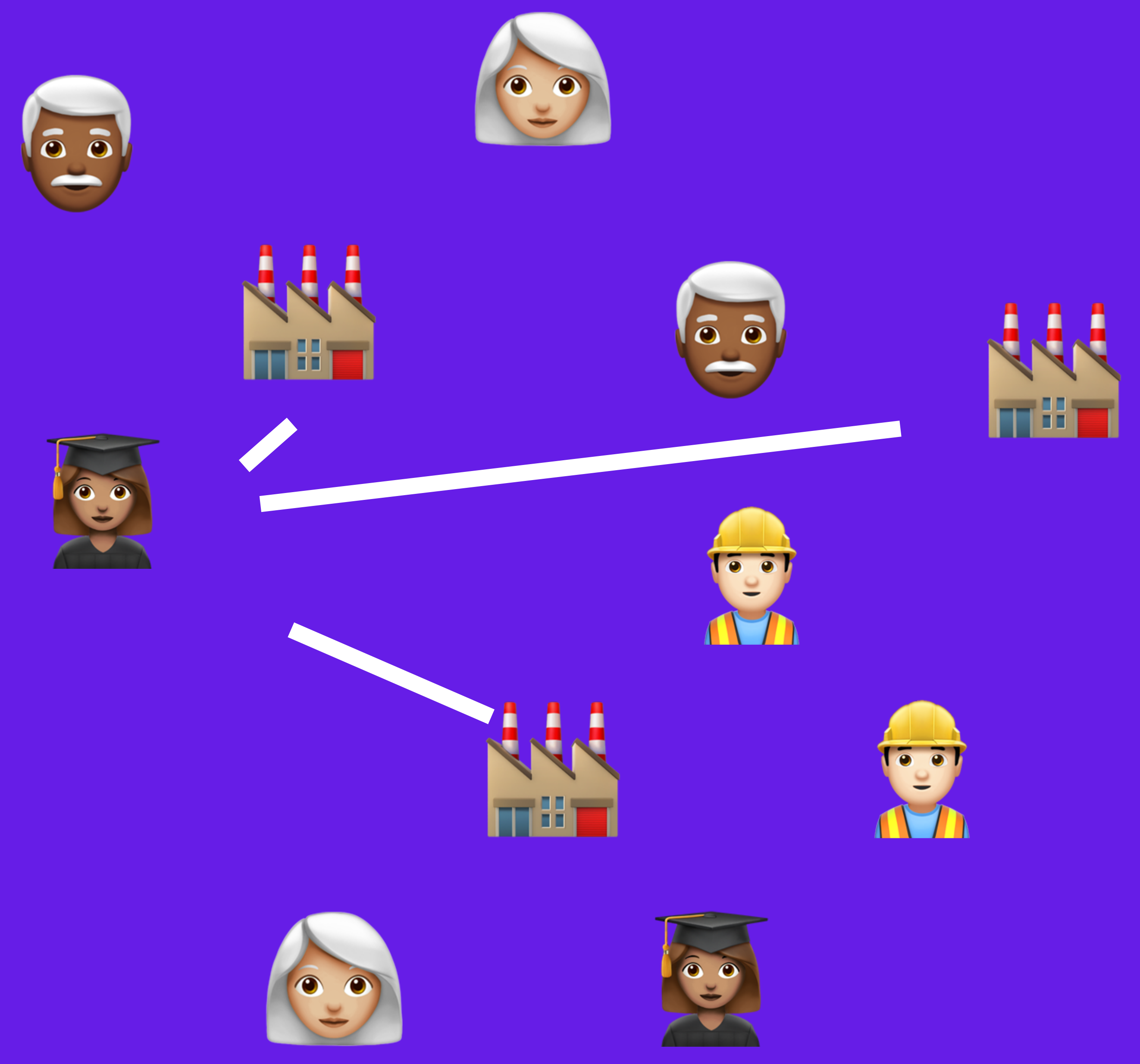
$$\max_{x^2} \quad 4x_1^2 + 2x_2^2 - x_1^2 x_1^1 - x_2^2 x_2^1$$

$$\text{s.t.} \quad 2x_1^2 + 3x_2^2 \leq 4$$

$$x^2 \in \{0,1\}^2$$

Knapsack Games (Carvalho et al., 2022; ***D.*** *and Scatamacchia, 2022)*

Facility Location and Design Game



Aboolian et al. (2007),
Cronert and Minner (2020),

Sellers (players) compete for the demand of customers located in a given geographical area. Each player decides:

- **Where** to open its selling facilities
- **What** assortment to sell (i.e., what design)

$$\max_{x^i} \quad \sum_{j \in J} w_j \frac{\sum_{l \in L} \sum_{r \in R_l} u^i_{ljr} x^i_{lr}}{\sum_{k=1}^{n} \sum_{l \in L} \sum_{r \in R_l} u^k_{ljr} x^k_{lr}}$$

Share of customers' demand

$$\text{s.t.} \sum_{l \in L} \sum_{r \in R_l} f^i_{lr} x^i_{lr} \leq B^i,$$

Budget

$$\sum_{r \in R_l} x^i_{lr} \leq 1 \quad \forall l \in L,$$

One facility per location

$$x^i_{lr} \in \{0, 1\} \quad \forall l \in L, \forall r \in R_l.$$

Energy


We are trying to
save energy –
This blanket will
keep you warm…

Multiple followers
dependent

$$
\begin{aligned}
\max_{x} \quad & f(x, y^1, \ldots, y^n) \\
\text{s.t.} \quad & (x, y^1, \ldots, y^n) \in X \\
& \max_{y^k} g(x, y^k, y^{-k}), k = 1, \ldots, n \\
& \text{s.t.} (x, y^k, y^{-k})) \in Y^k
\end{aligned}
$$

SolarCorp Inc.

Simultaneous Game

Hydro Inc.

“Cournot Game”

Canada taxes and regulates the production

SolarCorp Inc.

Simultaneous Game

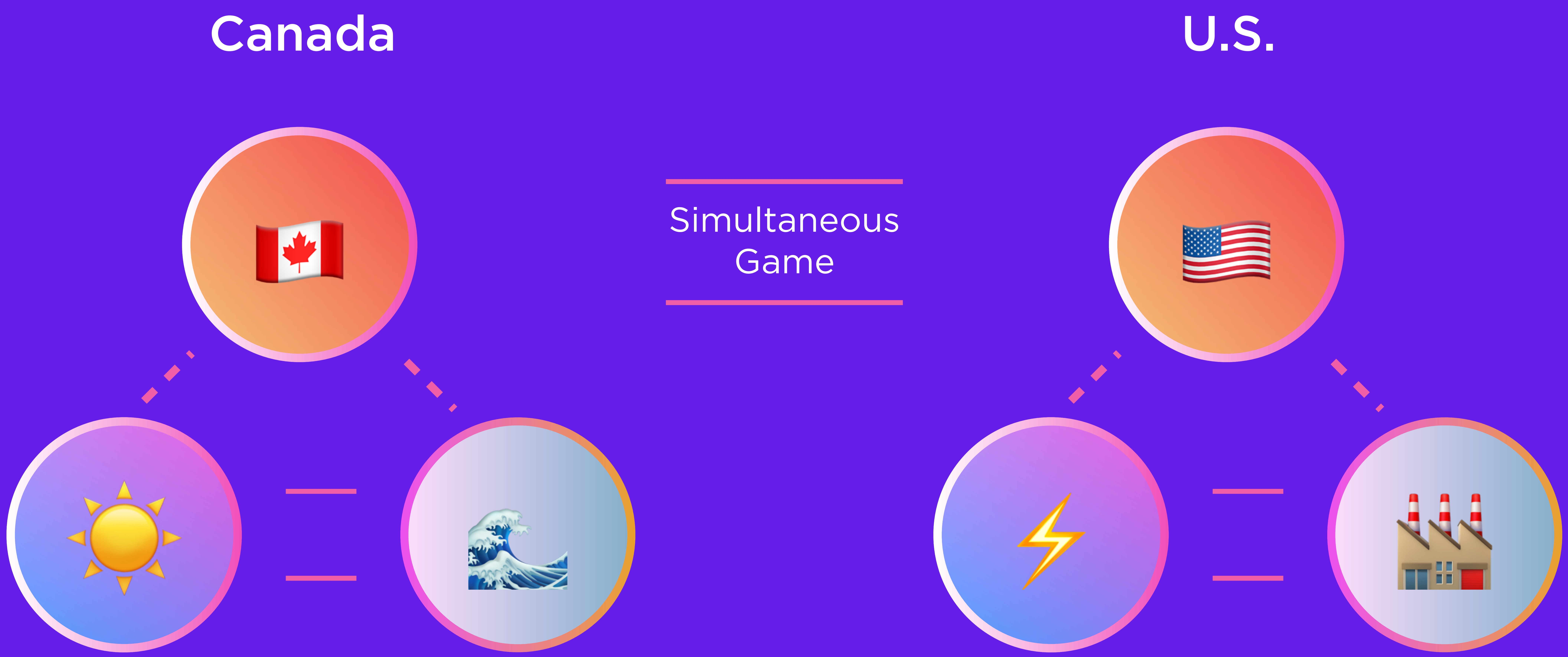
Hydro Inc.

Sequential
“Stackelberg” Game

Simultaneous
Game

SolarCorp Inc.

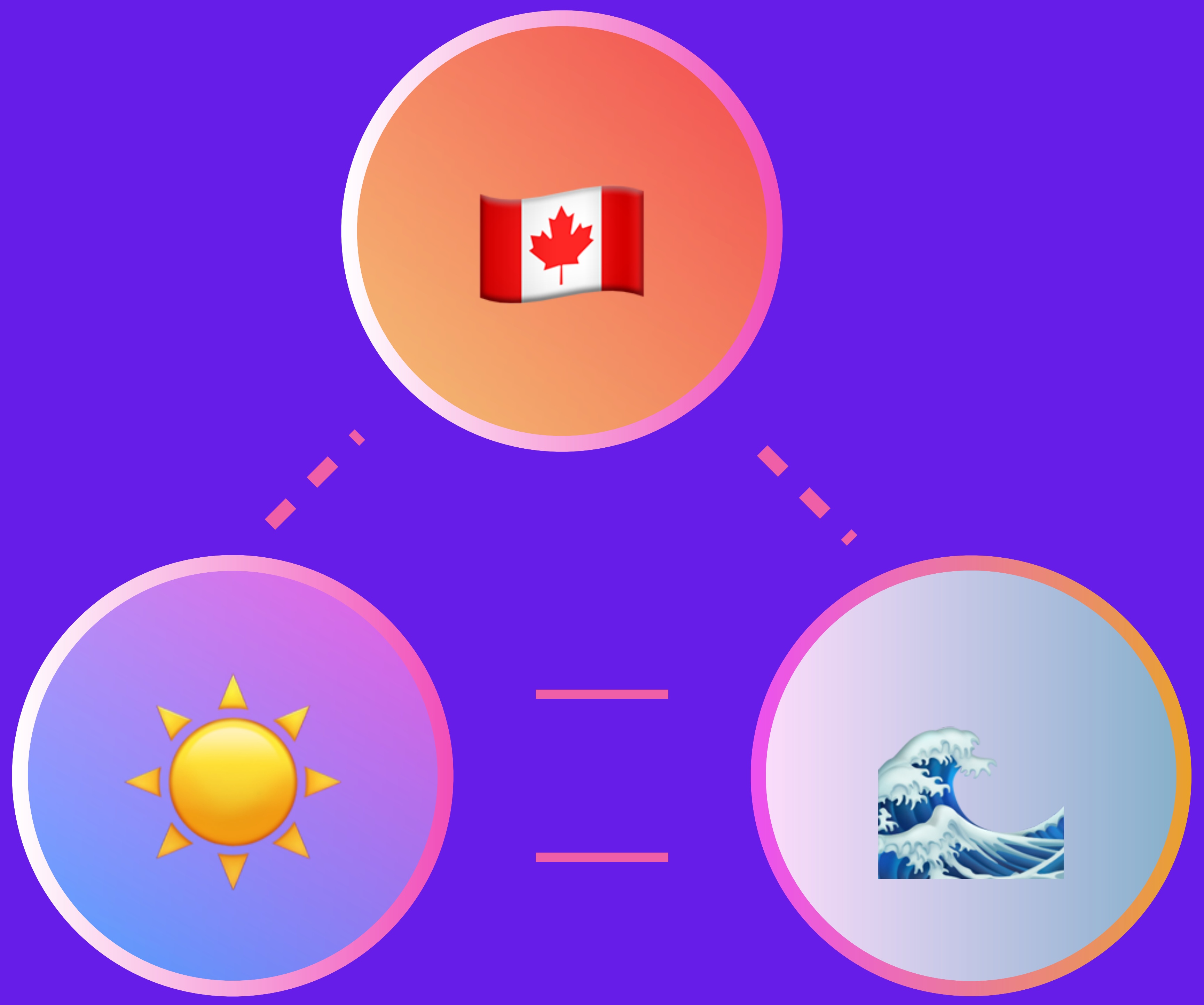
Hydro Inc.



This is a simultaneous game among **optimistic bilevel** (i.e., sequential) programs

When Nash Meets Stackelberg (Carvalho et al., 2022)

Canada



$$\max_{x^i}\{(c^i)^\top x^i + (x^{-i})^\top C^i x^i : x^i \in \mathscr{F}^i\}$$

The reformulated “Bilevel” feasible region includes the KKT for the followers’ problems @everybody

$$\left\{ \begin{array}{l} A^i x^i \leq b^i \\ z^i = M^i x^i + q^i \\ x^i \geq 0, z^i \geq 0 \end{array} \right\} \bigcap_{j \in \mathcal{C}^i} \left(\{z^i_j = 0\} \cup \{x^i_j = 0\}\right)$$

@Ted

@Martine's "Bilevel with Dependent Followers"

$$\max_{x^i}\{(c^i)^\top x^i + (x^{-i})^\top C^i x^i : x^i \in \mathscr{F}^i\}$$

$$\left\{ \begin{array}{l} A^i x^i \leq b^i \\ y^i \in \arg\max_{y^j} \{g^j(y^j, y^{-j}, w) : H^j y^j + K^j w \leq 0\} \quad \forall j = 1, \dots, J^i \\ x^i = (w, y^1, \dots, y^{J^i}) \end{array} \right\}$$

Each player i *solves a bilevel problem with:*

- *The leader having* **linear coupling constraints**
- The J^i *followers solving* **convex-quadratic problems** *parametrized in their leader and the other followers' variables*

What are IPGs?

What

Why

What are these games?

An ***Integer Programming Game (MPG)*** is a simultaneous one-shot (static) **game** among n players where each player $i = 1,\ldots,n$ solves

$$\max_{x^i} \{ u^i(x^i; x^{-i}) : x^i \in \mathcal{X}^i \}$$

Parametrized in $x^{-i} := (x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^n)$

The set of actions $\mathcal{X}^i$

$$\mathcal{X}^i := \{ A^i x^i \leq b^i, \; x^i \in \mathbb{Z}^{\alpha^i} \times \mathbb{R}^{\beta^i} \}$$

Köppe et al., (2011), Sagratella (2015), ***D.*** *et al (2021),* **D.** *(2022)*

Why IPGs?

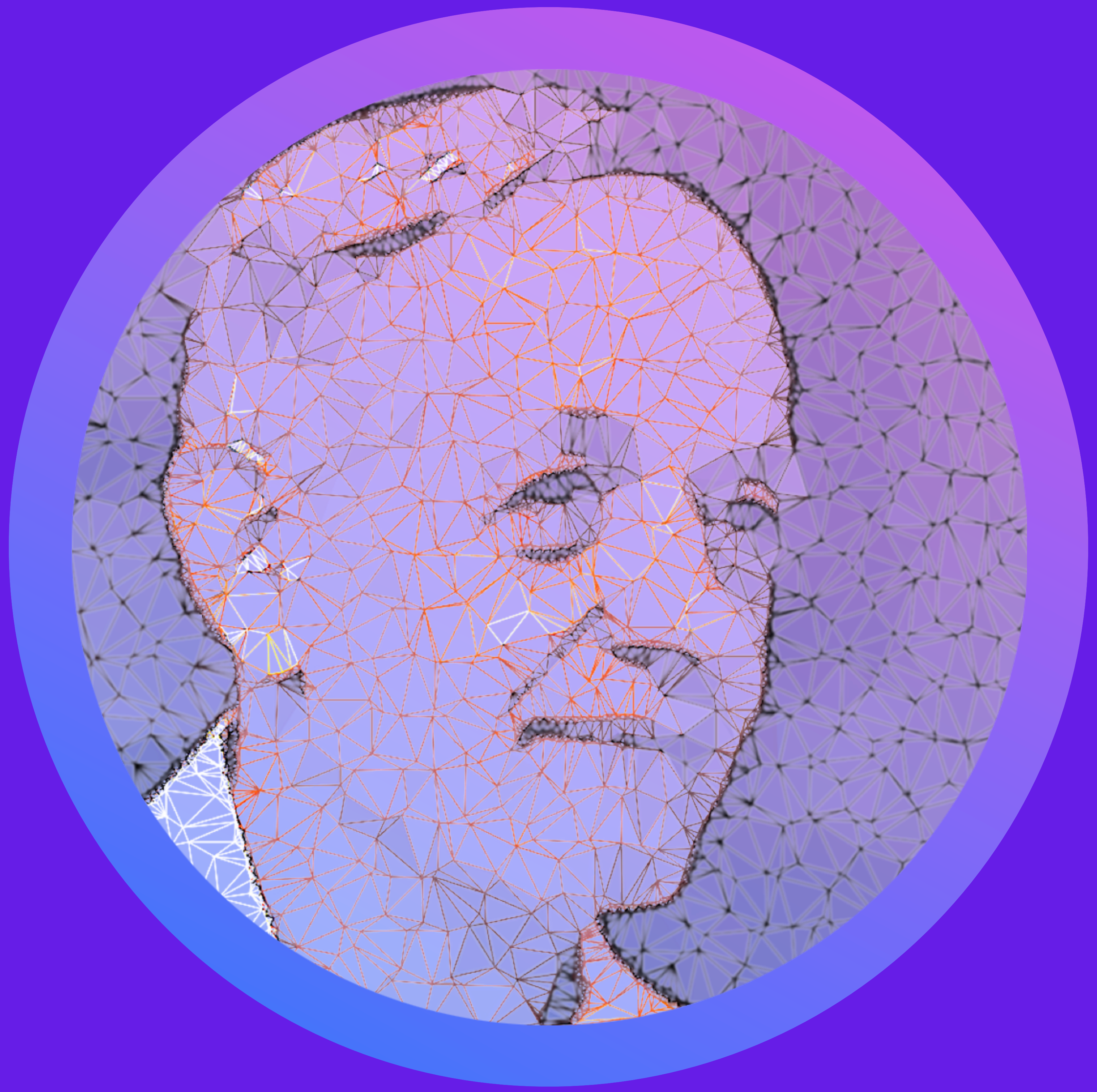
They extend traditional **resource-allocation tasks and combinatorial optimization** problems to a multi-agent setting

Indivisible quantities, **fixed production costs** and **logical disjunctions** often require discrete variables
(e.g., *Bikhchandani and Mamer (1997)*)

Energy — Gabriel et al. (2013), David Fuller and Çelebi (2017)
Supply Chain — Anderson et al. (2017)
Assortment-Price competitions — Federgruen and Hu (2015)
Kidney Exchange Problems — Carvalho et al. (2017)
Cybersecurity

+ A good stake of the people in this room

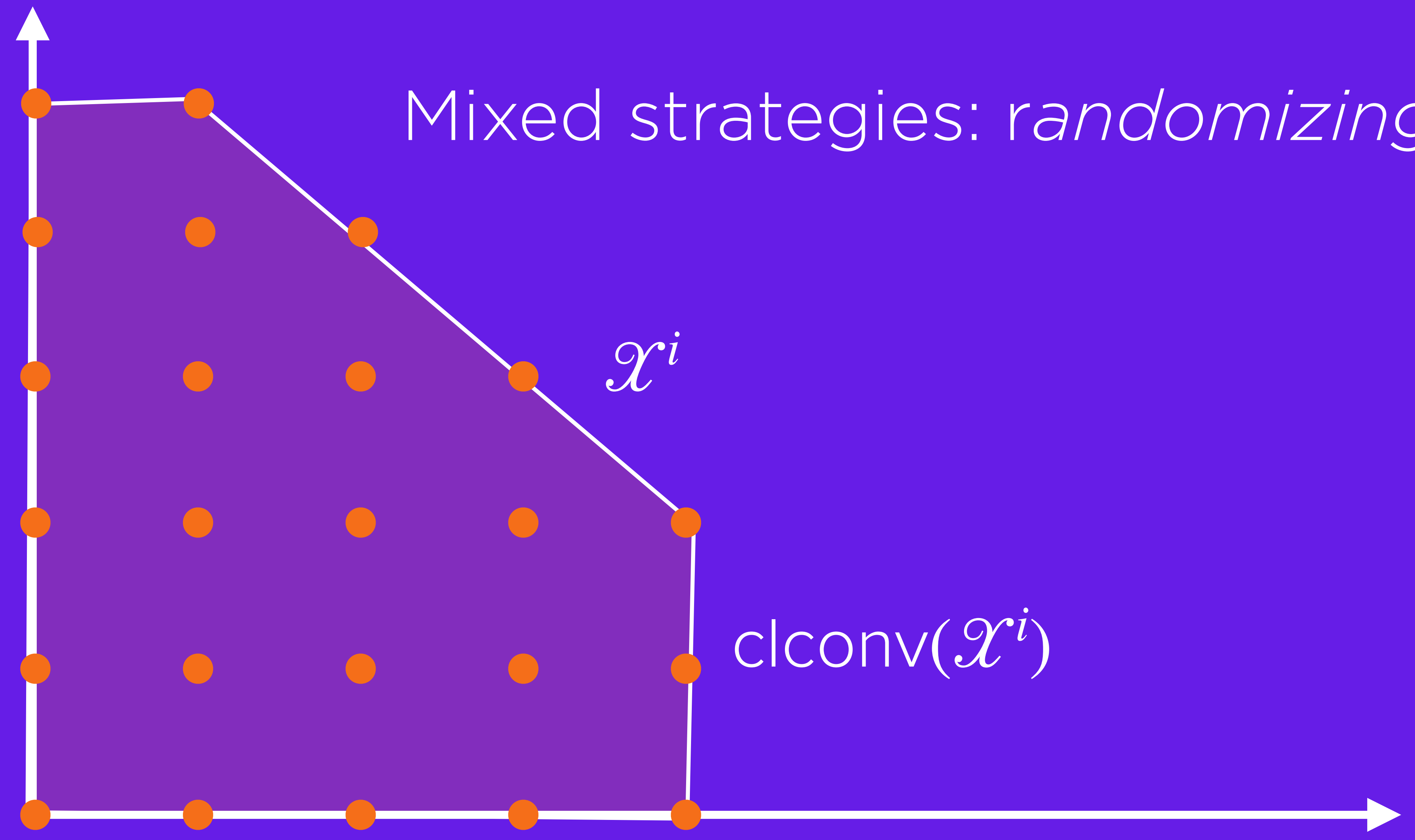
Nash Equilibria as Solutions



$\bar{x} = (\bar{x}^1, \ldots, \bar{x}^n)$ *is a Pure Nash Equilibrium (***PNE***) if*

$$u^i(\bar{x}^i, \bar{x}^{-i}) \geq u^i(\hat{x}^i, \bar{x}^{-i}) \quad \forall \hat{x}^i \in \mathscr{X}^i$$

Mixed strategies: *randomizing over clconv*$(\mathscr{X}^i)$



*Mixed Nash Equilibrium (***MNE***) if the above holds with mixed strategies*

Existence	When does at least an equilibrium exist?
Efficiency	How do different equilibria differ in their properties?
Algorithms	How do we compute and **select** equilibria?

Who's the specific case of whom?

Existence

How

Existence

*a.k.a.: the **second level***

How

Fundamental Theorems

PNEs and MNEs *(Carvalho et. al, 2018)*

1. Deciding if an IPG has a PNE is $\Sigma_2^p - complete$
2. Deciding if an IPG has a MNE is $\Sigma_2^p - complete$
3. Actually, if $\mathscr{X}^i$ is finite for any player i, there exists an MNE

The "Energy Game" *(Carvalho et. al, 2022)*

1. Deciding if an "Energy Game" has a PNE is $\Sigma_2^p - complete$
2. Deciding if an "Energy Game" has a MNE is $\Sigma_2^p - complete$
3. Actually, if $\mathscr{X}^i$ is finite for any player i, there exists an MNE

Fundamental Theorems

The "Energy Game" *(Carvalho et. al, 2022)*

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Knapsack Game *(D. and Scatamacchia, 2022)*

1. Deciding if a Knapsack Game has a PNE is $\Sigma_2^p - complete$

Efficiency

How

Their items **interact**!

$$\max_{x^1} \quad 6x_1^1 + x_2^1 - 4x_1^1x_1^2 + 3x_2^1x_2^2$$

$$\text{s.t.} \quad 3x_1^1 + 2x_2^1 \leq 4$$

$$x^1 \in \{0,1\}^2$$

$$\max_{x^2} \quad 4x_1^2 + 2x_2^2 - x_1^2x_1^1 - x_2^2x_2^1$$

$$\text{s.t.} \quad 3x_1^2 + 2x_2^2 \leq 4$$

$$x^2 \in \{0,1\}^2$$

Their items **interact**!

$$\max_{x^1} \quad 6x_1^1 + x_2^1 - 4x_1^1 x_1^2 + 3x_2^1 x_2^2$$

$$\text{s.t.} \quad 3x_1^1 + 2x_2^1 \leq 4$$

$$x^1 \in \{0,1\}^2$$

$$\max_{x^2} \quad 4x_1^2 + 2x_2^2 - x_1^2 x_1^1 - x_2^2 x_2^1$$

$$\text{s.t.} \quad 3x_1^2 + 2x_2^2 \leq 4$$

$$x^2 \in \{0,1\}^2$$

How *good* is a NE?

How good is a NE?

How good is a NE?

$$\max_{x^1} \quad 6x_1^1 + x_2^1 - 4x_1^1x_1^2 + 3x_2^1x_2^2$$
$$\text{s.t.} \quad 3x_1^1 + 2x_2^1 \leq 4$$
$$x^1 \in \{0,1\}^2$$

$$\max_{x^2} \quad 4x_1^2 + 2x_2^2 - x_1^2x_1^1 - x_2^2x_2^1$$
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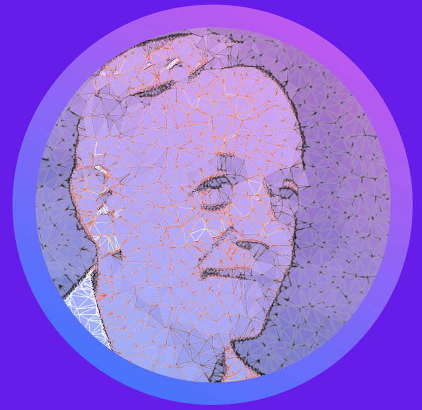
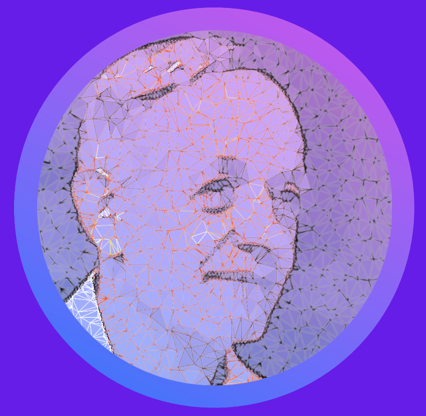
$$x^2 \in \{0,1\}^2$$

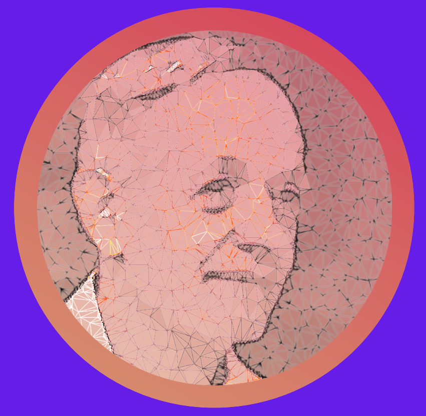


$(\bar{x}_1^1, \bar{x}_2^1) = (1,0)$ and $(\bar{x}_1^2, \bar{x}_2^2) = (1,0)$ with $W = 2 + 3 = 5$

$(\bar{x}_1^1, \bar{x}_2^1) = (1,0)$ and $(\bar{x}_1^2, \bar{x}_2^2) = (0,1)$ $W = 6 + 2 = 8$



$$= \frac{\text{Optimal Social Welfare}}{\text{"Best" NE}} = PoS$$



$$= \frac{\text{Optimal Social Welfare}}{\text{"Worst" NE}} = PoA$$

Algorithms

How

How?

How do we use and solve them in practice

ZERO Regrets

Optimizing over equilibria in **Integer Programming Games**

(**D**. and Scatamacchia, 2021)

Cut-And-Play

Computing Nash equilibria via **Convex Outer Approximations**

(Carvalho et al., 2021)

How?

How do we use and solve them in practice

ZERO Regrets

The “Robust” way

Cut-And-Play

The “Dual” Bilevel way

The ZERO Regrets Algorithm

Joint work with **Rosario Scatamacchia** (Politecnico di Torino, Italy)

How

Integer Programming Games

We consider Pure-Integer IPGs with bounded variables
(although this generalizes to mixed-integer)

$$\max_{x^i} \{ u^i(x^i, x^{-i}) : x^i \in \mathcal{X}^i \}, \ \mathcal{X}^i := \{ A^i x^i \leq b^i, x^i \in \mathbb{Z}^m \}$$

There is **common knowledge of rationality**, thus each player is **rational**
and there is **complete information**,

Selection

Not all Nash equilibria were created equal
i.e., **Price of Stability (PoS) and Anarchy (PoA)**

Tractability

Existence

Restrictive assumptions on the game's structure to guarantee the existence/tractability

Methodology

Lack of a general-purpose methodology to compute and mostly **select** equilibria

⇒ **No general methodology, no broad use of IPGs.**

Type of NE

	General	Enumer.	Select	PNE	NE	Approx	Limitations
ZERO Regrets	✅	✅	✅	✅	✅	✅	**Most efficient, selection, existence, enumeration**
Koeppe et al. (2011)	✅	✅	❌	✅	❌	❌	No (practical) algorithm
Sagratella (2016)	✅	✅	❌	✅	❌	❌	Convex payoffs
Del Pia et al. (2017)	❌	❌	❌	✅	❌	❌	Problem-specific (unimodular)
Carvalho, D., Lodi, Sankaranarayanan (2020)	✅	❌	❌	❌	✅	❌	Bilinear payoffs
Cronert and Minner (2021)	✅	✅	❌	❌	✅	❌	No selection, expensive, existence?
Carvalho et al. (2022)	✅	❌	❌	❌	✅	✅	No selection/enumeration, existence?
Schwarze and Stein (2022)	✅	✅	❌	✅	❌	❌	Expensive Branch-and-Prune

Lack of a **general-purpose methodology** to compute and mostly select equilibria

Our Goal

Given an IPG, compute the Nash equilibrium maximizing a function $f(x^1, \ldots, x^n)$

High-Level Idea

1. **Initialization**

$$\mathcal{K} = \{(x,z) : x \in \prod_i \mathcal{X}^i, (x,z) \in \mathcal{L}\} \qquad \Phi := \{0 \leq 1\}$$

2. **Optimization**

$$\bar{x} = \arg \max_{x^1,\ldots,x^n,z} \{f(x,z) : (x,z) \in \mathcal{K}, (x,z) \in \Phi\}$$

3. **Separation**

$$\tilde{x}^i = \arg \max_{x^i} \{u^i(x^i, \bar{x}^{-i}) : A^i x^i \leq b^i, x^i \in \mathbb{Z}^m\}$$

If there is a player i so that $u^i(\tilde{x}^i, \bar{x}^{-i}) \geq u^i(\bar{x}^i, \bar{x}^{-i})$

Then, $\Phi = \Phi \cup \{\ u^i(\hat{x}^i, x^{-i}) \leq u^i(x^i, x^{-i})\ \}$ and goto 2

Else: $\bar{x}$ is the PNE maximizing f

Why does it work

Equilibrium Inequality

An inequality is an **equilibrium inequality** *if it is valid for* $\mathcal{E}$*, i.e., the set of Nash equilibria*

$$u^i(\tilde{x}^i, x^{-i}) \leq u^i(x^i, x^{-i}) \quad \forall \tilde{x}^i \in \mathcal{X}^i$$

*suboptimal @Ivana

Theorem *(D. and Scatamacchia, 2022)*

$$P^e := \operatorname{conv}\left(\left\{ (x,z) \in \mathcal{K} : \begin{array}{l} u^i(\tilde{x}^i, x^{-i}) \leq u^i(x^i, x^{-i}) \\ \forall \tilde{x} : \tilde{x}^i \in \mathcal{BR}(i, \tilde{x}^{-i}),\ i = 1, \dots, n \end{array} \right\} \right)$$

(1) P^e **is a polyhedron**

(2) $\nexists (x,z) \in P^e : x \in \mathbb{Z}^{nm}$

(3) $P^e = \mathcal{E}$

Why does it work?

Why does it work

$$u^i(\tilde{x}^i, x^{-i}) \leq u^i(x^i, x^{-i}) \quad \forall \tilde{x}^i \in \mathcal{X}^i$$

Let's generalize:

Assume each player i solves:

$$\max_{x^i} \quad f^i(x^i, x^{-i})$$
$$s.t. \quad x^i \in \mathcal{X}^i$$

f^i concave in x^i

Nash Equilibria $\Longrightarrow$

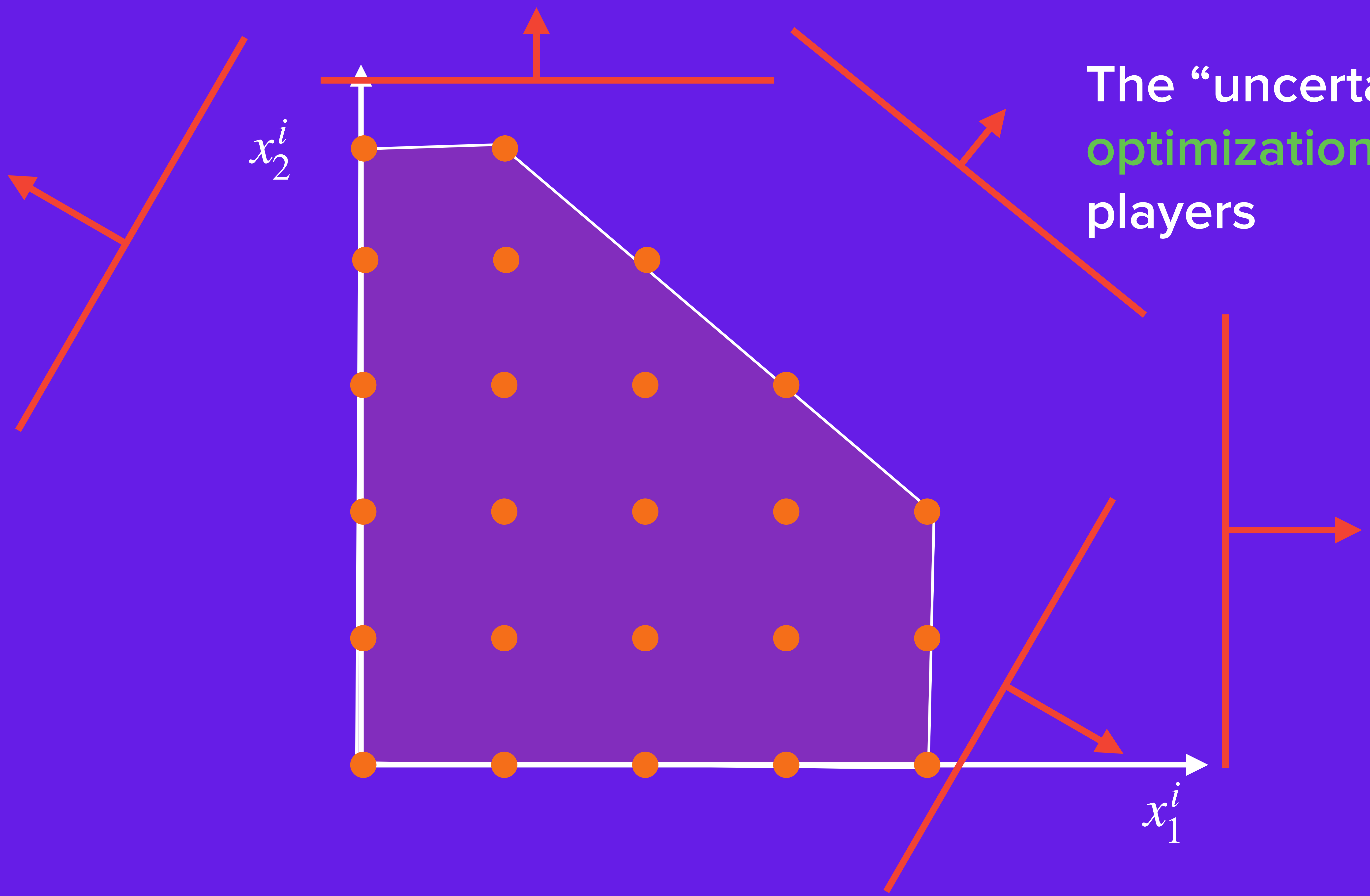
$$x^i \in \mathcal{X}^i \quad \forall i = 1, \ldots, n$$
$$f^i(\tilde{x}^i, x^{-i}) \leq f^i(x^i, x^{-i}) \quad \forall \tilde{x}^i \in \mathcal{X}^i \quad \forall i = 1, \ldots, n$$

In the IPG case, **"polyhedral" uncertainty** on the ***convex-hull of the integer solutions ofeach player***

"The Trouble with the Second Quantifier"

$$u^i(\tilde{x}^i, x^{-i}) \leq u^i(x^i, x^{-i}) \quad \forall \tilde{x}^i \in \mathcal{X}^i$$

In the IPG case, **"polyhedral" uncertainty** wrt the ***integer convex-hull of each player***



The "uncertainty" is the optimization problem of the other players

The Trouble with the Second Quantifier

$$u^i(\tilde{x}^i, x^{-i}) \leq u^i(x^i, x^{-i}) \quad \forall \tilde{x}^i \in \mathcal{X}^i$$

In the IPG case, **"polyhedral" uncertainty** on the ***integer convex-hull of each player***

Alternative Proof:

- binary problems with binary uncertainty ($\{0,1\}^n$ intersected with polyhedron) are Σ_2^p-hard [CS20]

@Marc

Let’s Generalize More

Why does it work

Assume each player i solves:

$$\max_{x^i} \quad f^i(x^i, x^{-i})$$
$$s.t. \quad x^i \in \mathcal{X}^i(x^{-i})$$

f^i concave in x^i

$\mathcal{X}^i$ parametrized in x^{-i}

Generalized Nash Equilibria $\Longrightarrow$

Currently working on it… but this looks like…

Decision-dependent Uncertainty?

Applications

	Applications	Baselines	Select	Enumer.	Improvement
Knapsack Game	Packing, Assortment Optimization	Carvalho et al. (2021, 2022)	❌	❌	N.A.
Network Formation Games	Network design, the Internet, cloud infrastructure	Chen and Roughgarden (2006), Anshelevich, et al. (2008), Nisan et al. (2008)	✅	❌	N.A.
Facility Location Games	Retail, cloud service provisioning	Cronert and Minner (2021)	✅	❌	>50x

Knapsack Game (*KPG*)

As for Wizard and Fairy, each player solves a binary Knapsack problem with some **interaction terms** in the objective

$$\max_{x^i} \left\{ \sum_{j=1}^{m} p_j^i x_j^i + \sum_{k=1, k\neq i}^{n} \sum_{j=1}^{m} C_{k,j}^i x_j^i x_j^k : \sum_{j=1}^{m} w_j^i x_j^i \leq b^i, \mathbf{x}^i \in \{0,1\}^m \right\}$$

Knapsack Game (*KPG*)

A few facts:

- No successful attempts to **enumerate or select** equilibria in KPGs with $n > 2$ and $m > 4$ (*Cronert and Minner (2021)*)
- Carvalho et al. (2021, 2022) only compute **an MNE** with at most $n = 3, m \leq 40$
- No results on the complexity of the KPG, nor its PoS/PoA

We select PNEs with $n > 2, m > 50$
We provide "packing" equilibrium inequalities

We prove it is Σ_2^p-complete to determine if a PNE exists + the PoS/PoA are arbitrarily bad

Knapsack Game (*KPG*)

Equilibrium inequalities may also **capture specific structures** or constraint types.

Strategic Payoff Inequalities

A fact

In a packing problem, often the all-zeros strategy is feasible with objective 0

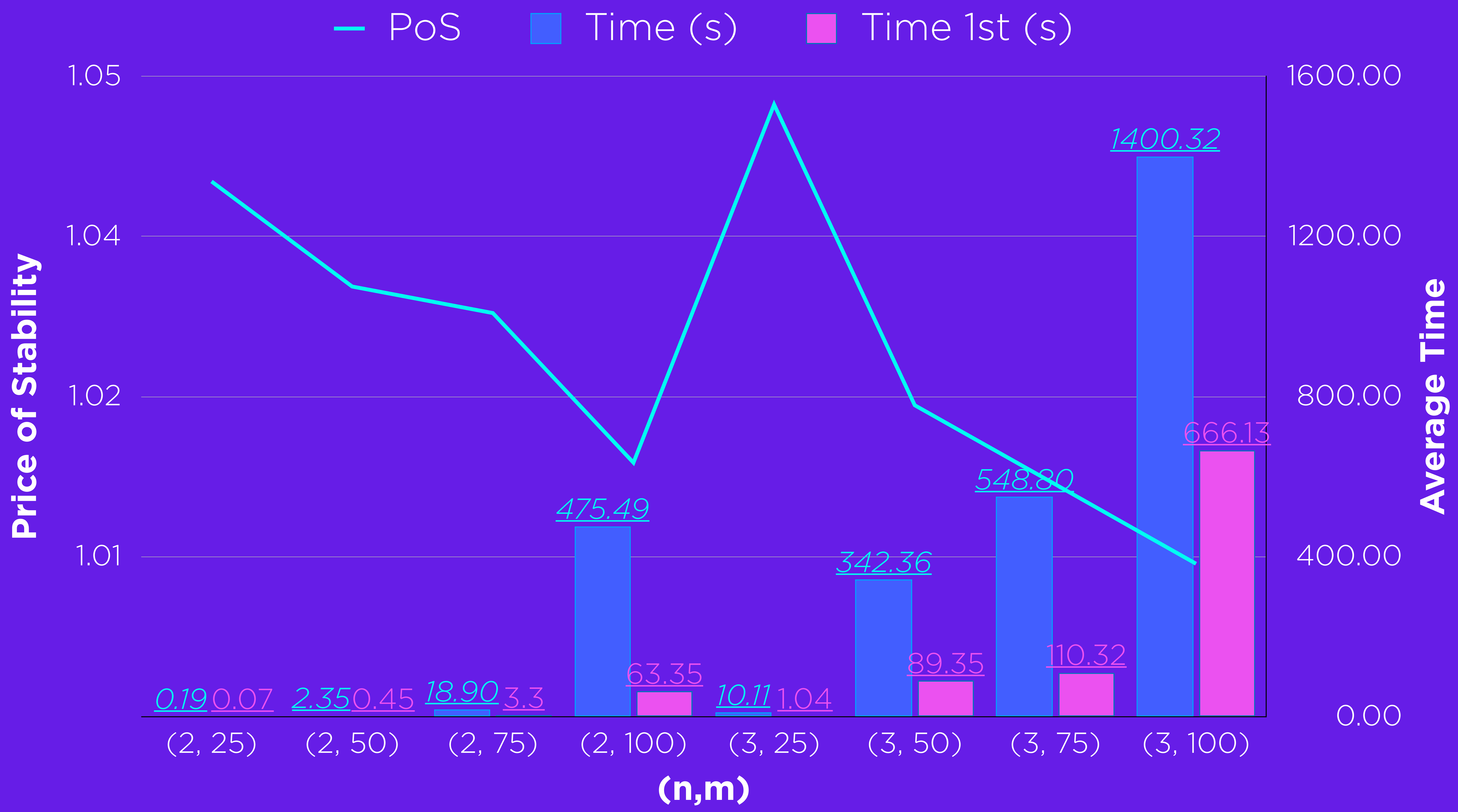
A consequence

Let $\mathcal{S}_i$ be a subset of i's opponents. If $\exists \mathcal{S}_i$ so that

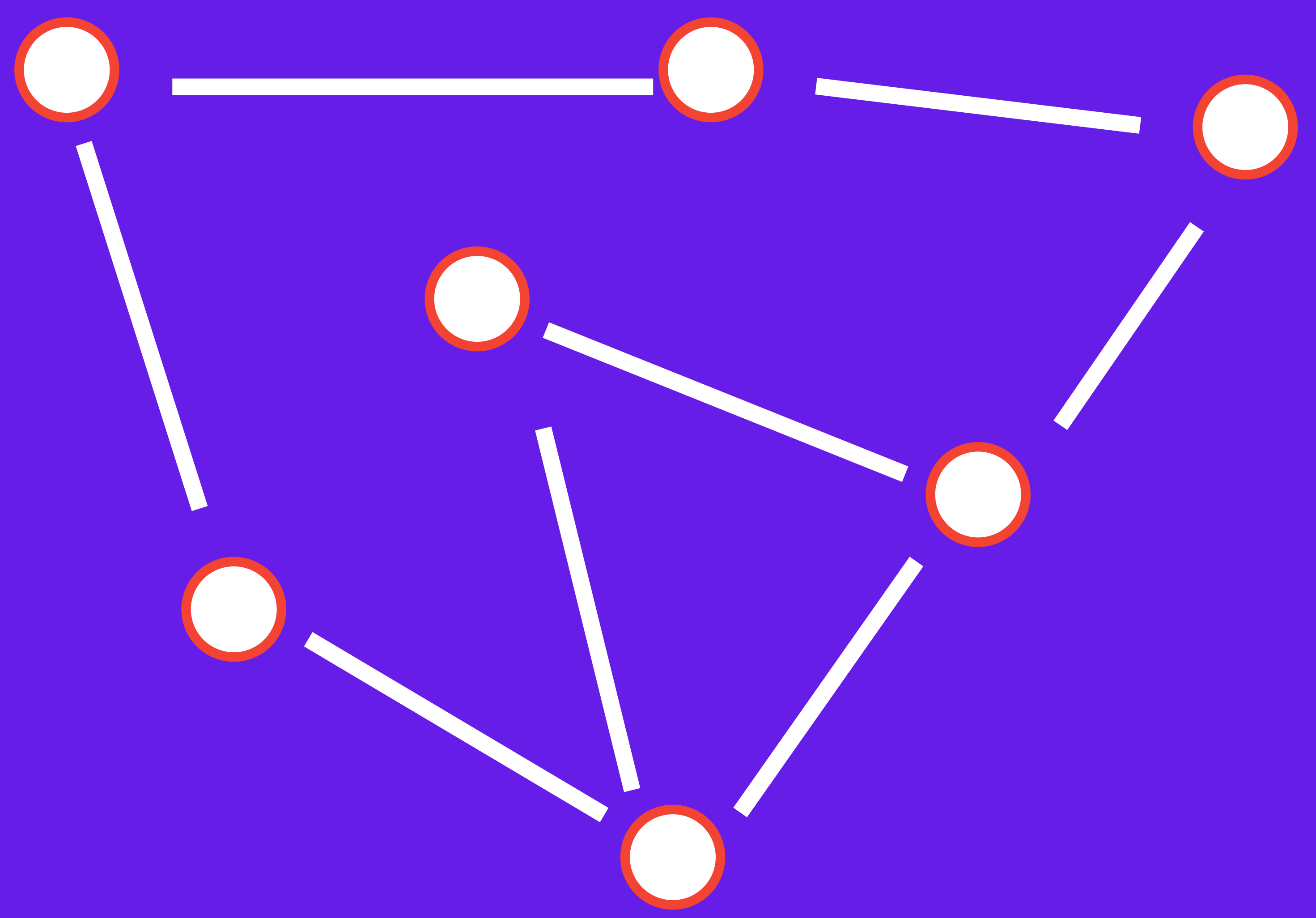
$$p^i_j + \sum_{k \in \mathcal{S}^i_j} C^i_{k,j} < 0,$$

then, $x^i_j + \sum_{k \in \mathcal{S}^i_j} x^k_j \leq |\mathcal{S}^i_j|$ is an **equilibrium inequality**.

Knapsack Game



Network Formation Game



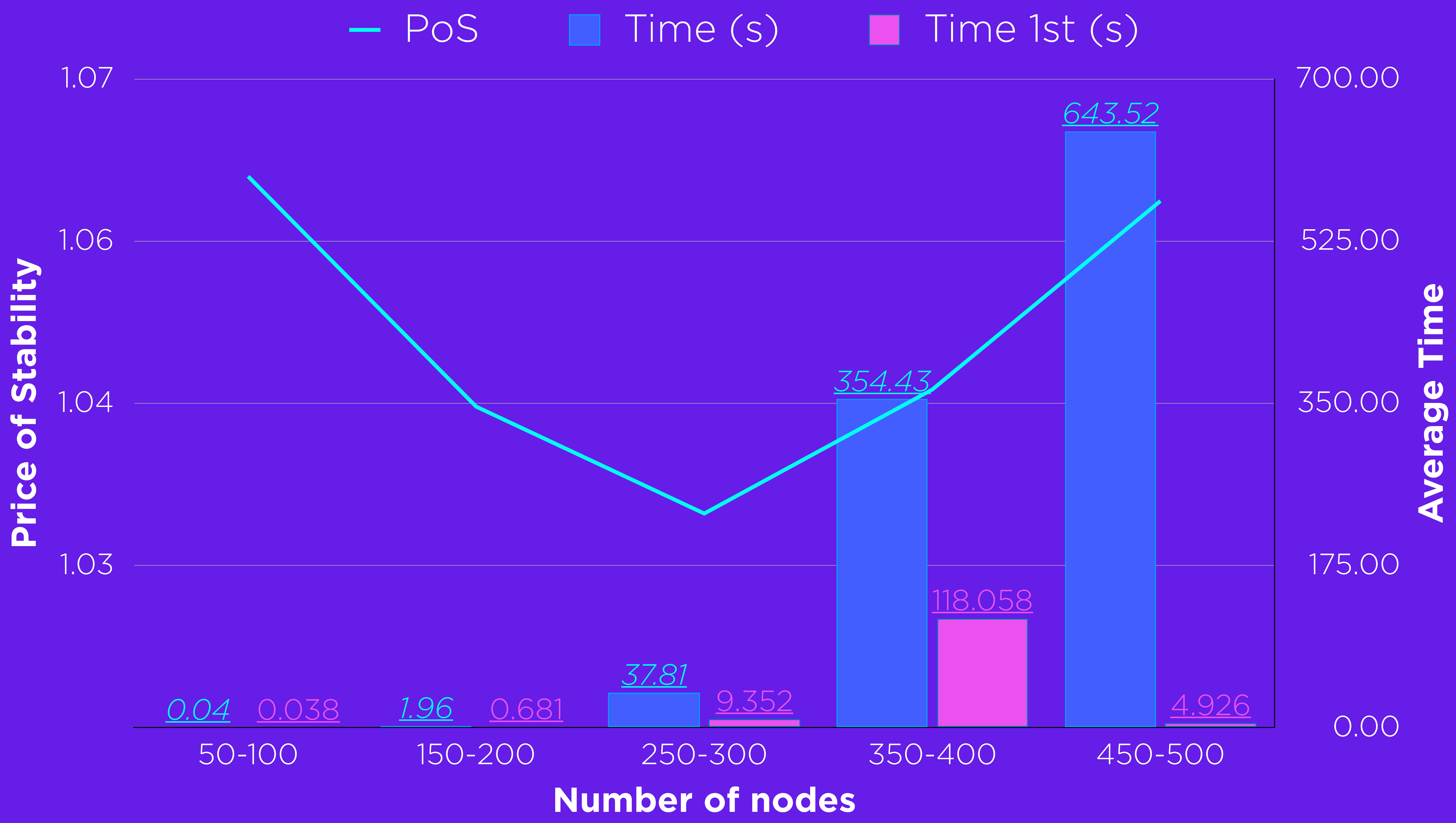
$G = (V, E)$

$$\min_{x^i} \{ \sum_{(h,l) \in E} \frac{w^i c_{hl} x^i_{hl}}{\sum_{k=1}^n w^k x^k_{hl}} : x^i \in \mathcal{X}^i \}.$$

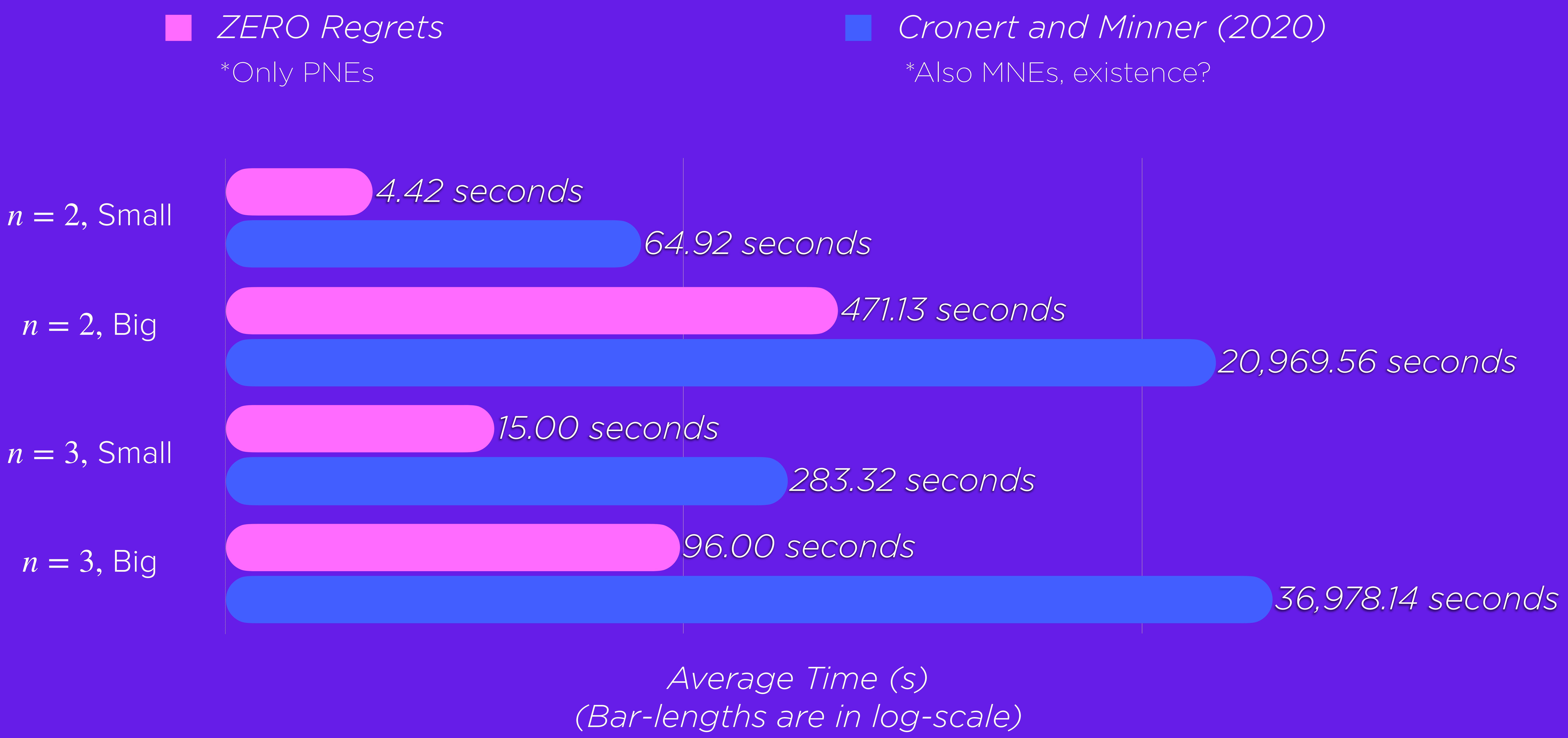
A few facts:

- No algorithms to **select** equilibria in arbitrary NFGs
- Several bounds on PoS/PoA in some specific instances
- We consider the **weighted version** with $n = 3$

Network Formation Game



Facility Location and Design Game



Rationality

Thanks to Claudio Sole*

Rationality

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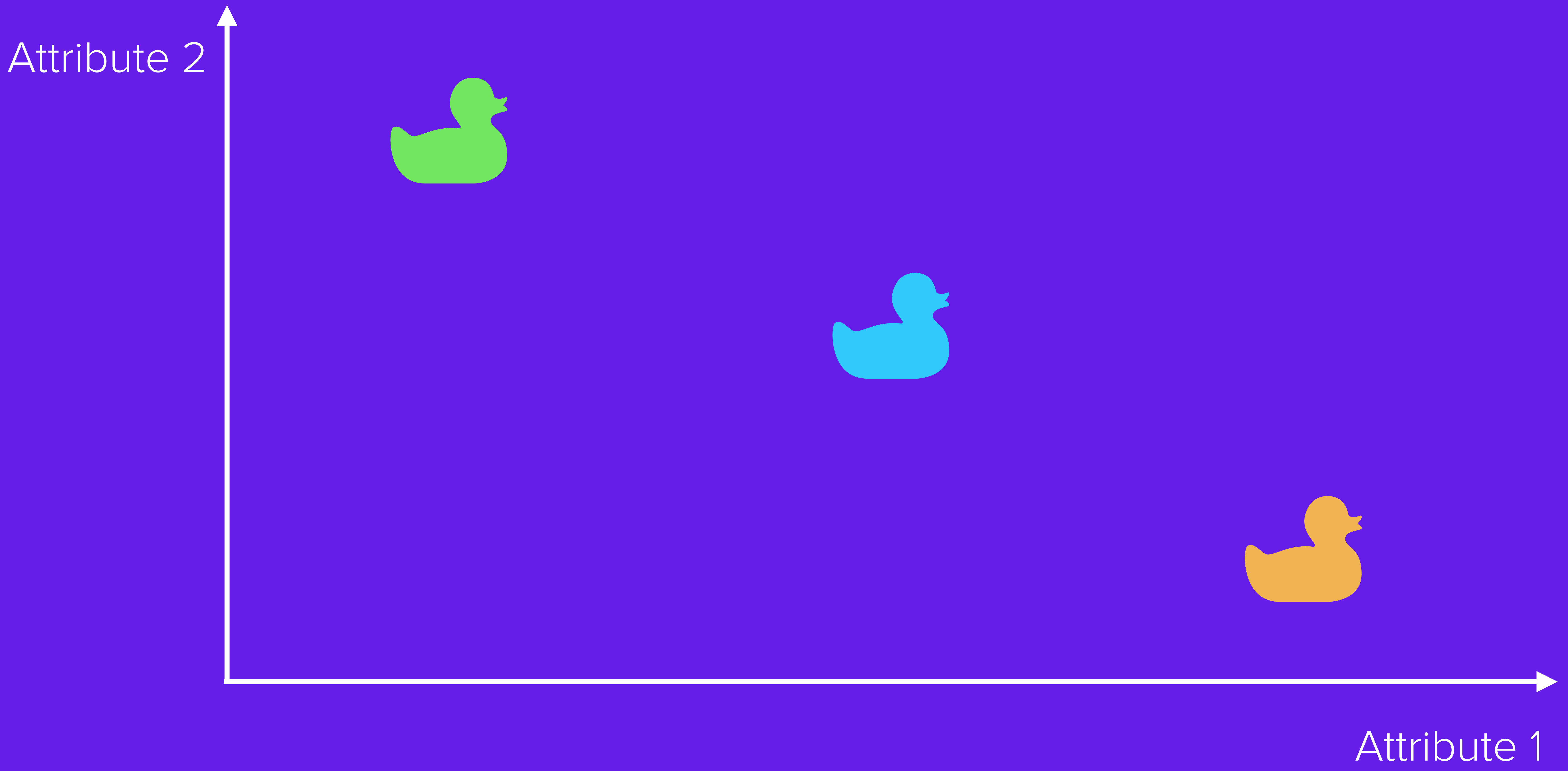
Rationality



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