Integer Programming Games

A Gentle Computational Overview

Margarida Carvalho, **Gabriele Dragotto, Andrea Lodi,** and Sriram Sankaranarayanan

informs
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CANADA EXCELLENCE RESEARCH CHAIR DATA SCIENCE FOR REAL-TIME DECISION-MAKING
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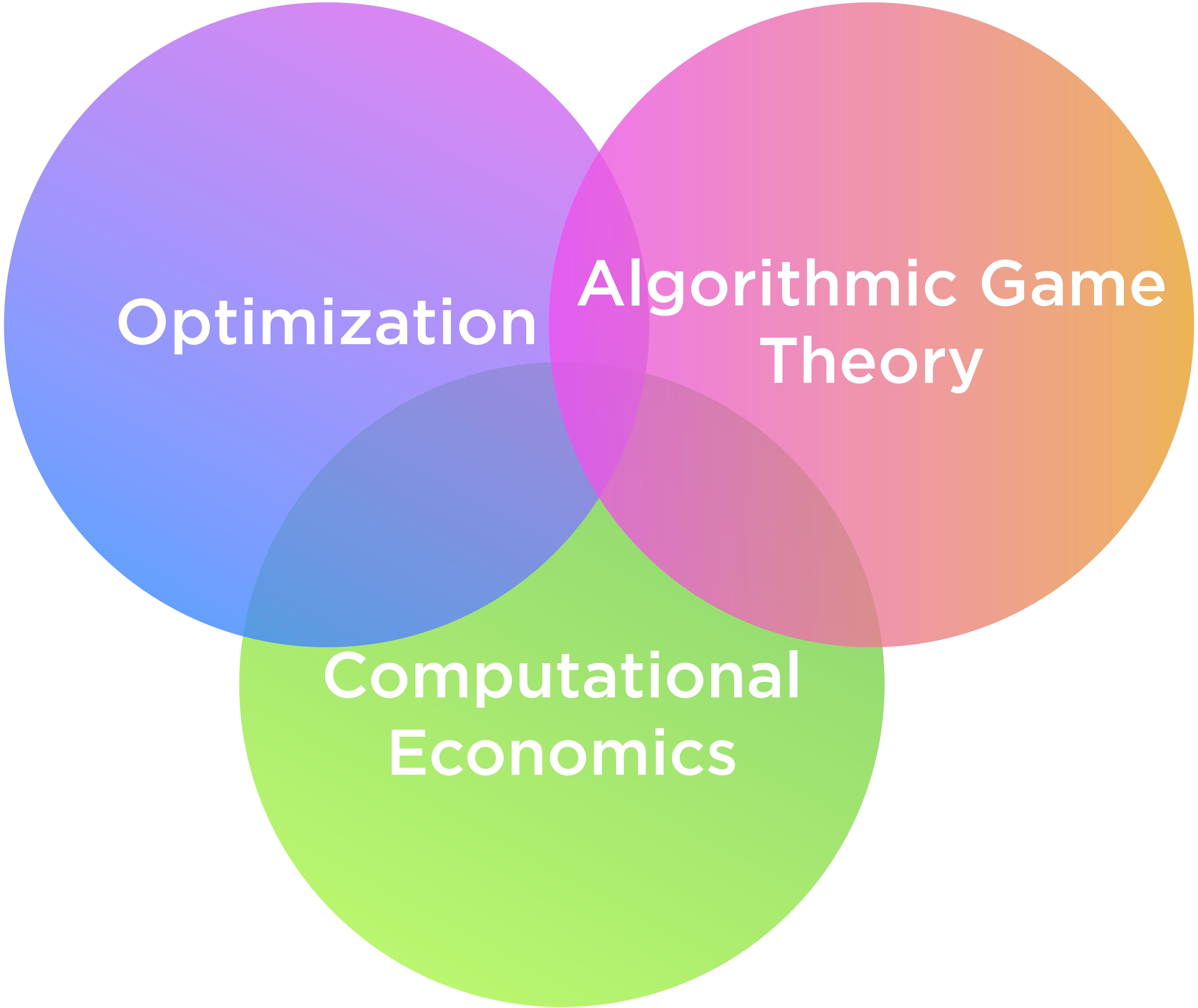
Optimization
Algorithmic Game Theory
Computational Economics

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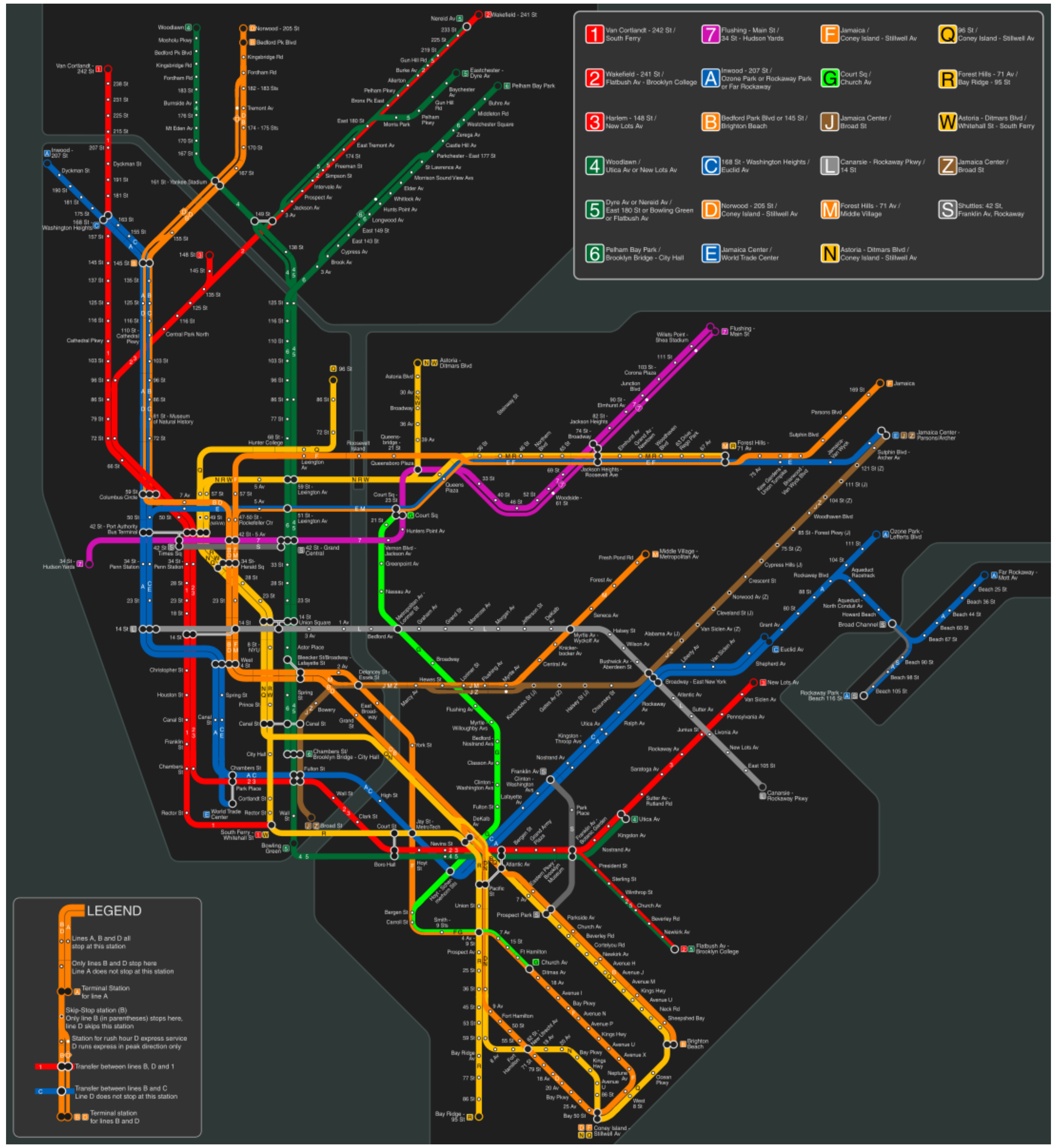
What are IPGs?

Commuting

Congestion Games



There are n players simultaneously optimizing the **shortest path** on a network

Choices of **other players**

$$\min_{x^i}\{u^i(x^i; x^{-i}) : x^i \in \mathcal{X}^i\}$$

Choices of **player i**

How do we **algorithmically compute** a ***stable* outcome?**

Multi-agent Assortment

$$
\begin{aligned}
\underset{x^1}{\text{maximize}} \quad & x_1^1 + 2x_2^1 \\
\text{s.t.} \quad & 3x_1^1 + 4x_2^1 \leq 5, \\
& x^1 \in \{0,1\}^2
\end{aligned}
$$

$$
\begin{aligned}
\max_{x^1} \quad & x^1_1 + 2x^1_2 \\
\text{s.t.} \quad & 3x^1_1 + 4x^1_2 \le 5, \\
& x^1 \in \{0,1\}^2
\end{aligned}
$$

Their "profits" **interact**

$$\begin{aligned} \underset{x^1}{\text{maximize}} \quad & x^1_1 + 2x^1_2 \\ \text{s.t.} \quad & 3x^1_1 + 4x^1_2 \leq 5, \\ & x^1 \in \{0,1\}^2 \end{aligned}$$

$$\begin{aligned} \underset{x^2}{\text{maximize}} \quad & 3x^2_1 + 5x^2_2 \\ \text{s.t.} \quad & 2x^2_1 + 5x^2_2 \leq 5, \\ & x^2 \in \{0,1\}^2 \end{aligned}$$

*Knapsack Games (**C.** et al., 2022; **D.** and Scatamacchia, 2022), Critical Node Game (**D.** et al., 2023)*

Their "profits" **interact**

$$\begin{aligned}\max_{x^1}\ & x^1_1 + 2x^1_2 - 2x^1_1x^2_1 - 3x^1_2x^2_2\\ \text{s.t.}\ & 3x^1_1 + 4x^1_2 \le 5,\\ & x^1 \in \{0,1\}^2\end{aligned}$$

$$\begin{aligned}\max_{x^2}\ & 3x^2_1 + 5x^2_2 - 5x^2_1x^1_1 - 4x^2_2x^1_2\\ \text{s.t.}\ & 2x^2_1 + 5x^2_2 \le 5,\\ & x^2 \in \{0,1\}^2\end{aligned}$$

*Knapsack Games (**C.** et al., 2022; **D.** and Scatamacchia, 2022), Critical Node Game (**D.** et al., 2023)*

Stable solutions

$$\underset{x^1 \in \{0,1\}^2}{\text{maximize}} \quad x_1^1 + 2x_2^1 - 2x_1^1 x_1^2 - 3x_2^1 x_2^2$$
$$\text{s.t.} \quad 3x_1^1 + 4x_2^1 \leq 5,$$

$$\underset{x^2 \in \{0,1\}^2}{\text{maximize}} \quad 3x_1^2 + 5x_2^2 - 5x_1^2 x_1^1 - 4x_2^2 x_2^1$$
$$\text{s.t.} \quad 2x_1^1 + 5x_2^1 \leq 5,$$

x_1^1	x_2^1	x_1^2	x_2^2		
0	0	0	0	0	0
1	0	0	0	1	0
0	1	0	0	2	0
0	0	1	0	0	3
0	0	0	1	0	5
1	0	1	0	-1	-2
0	1	0	1	-1	-1
0	1	1	0	2	3
1	0	0	1	1	5

Three **feasible strategies** for each player:

$$(x_1^i, x_2^i) \in \{(0,0), (0,1), (1,0)\}$$

But **only two** guarantee **stability**

and

Stable solutions

$$\underset{x^1\in\{0,1\}^2}{\text{maximize}} \quad x_1^1 + 2x_2^1 - 2x_1^1x_1^2 - 3x_2^1x_2^2 \qquad \text{s.t.} \quad 3x_1^1 + 4x_2^1 \le 5,$$

$$\underset{x^2\in\{0,1\}^2}{\text{maximize}} \quad 3x_1^2 + 5x_2^2 - 5x_1^2x_1^1 - 4x_2^2x_2^1 \qquad \text{s.t.} \quad 2x_1^1 + 5x_2^1 \le 5,$$

🍎 x_1^1	🧀 x_2^1	🍏 x_1^2	🍫 x_2^2	blue	red
0	0	0	0	0	0
1	0	0	0	1	0
0	1	0	0	2	0
0	0	1	0	0	3
0	0	0	1	0	5
1	0	1	0	-1	-2
0	1	0	1	-1	-1
0	1	1	0	2	3
1	0	0	1	1	5

But **only two** guarantee **stability**

🍎 🍫

Players **cannot profitably deviate**:

- If **blue** plays 🧀, it would get **-1** instead of **2**
- If **red** plays 🍫, it would get **-1** instead of **3**

Energy Markets

SolarCorp Inc.

Simultaneous Game

Hydro Inc.

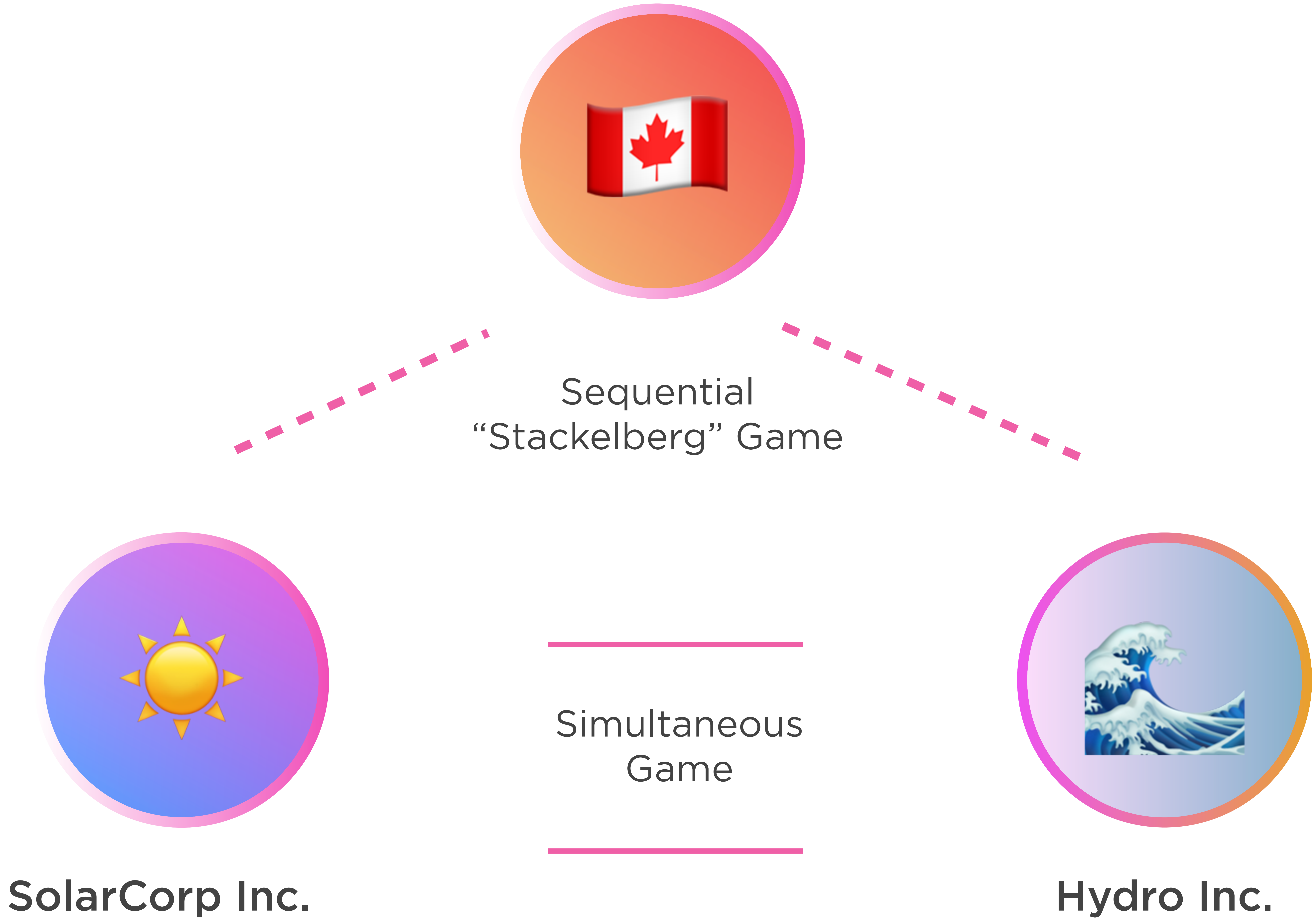
“Cournot Game”

Canada taxes and regulates the production

SolarCorp Inc.

Simultaneous
Game

Hydro Inc.


Sequential
“Stackelberg” Game
Simultaneous
Game
SolarCorp Inc.
Hydro Inc.

This is a simultaneous game among **bilevel** (i.e., sequential) programs

*When Nash Meets Stackelberg (**C.** et al., 2022)*

And it can get more complex…

And it get more complex…

Supply Chain and Transportation

Cronert and Minner, 2021 (OR, TR-B)

Sagratella et al., 2020 (EJOR)

***C.** et al., 2018 (IJ Production Economics)*

Simultaneous game among “bilevel” players

***C.** et al., 2023*

(Management Science)

Cybersecurity

***D.** et al., 2023*

(Ericsson Inc, - Patent pending)

Decision-making is rarely an individual task

It involves the mutual interaction of several self-interested agents and their individual preferences

Decision-making is rarely an individual task

What if agents (players) decide by solving mixed-integer optimization problems?

The Toolkit: Integer Programming Games

An Integer Programming Game (IPG) is a **simultaneous one-shot (static)** game among n players where each player $i = 1, \ldots, n$ solves

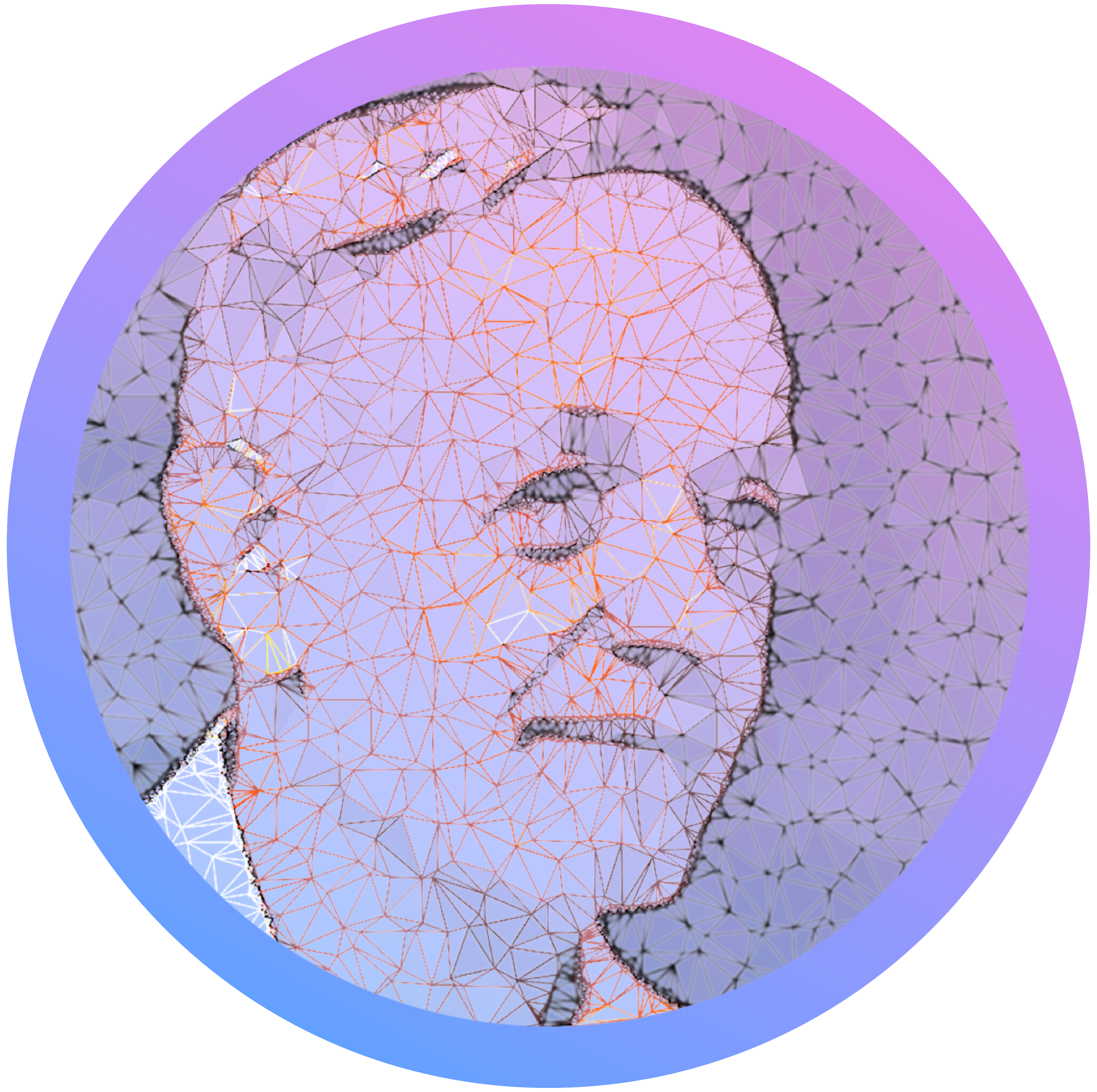
$$\max_{x^i} \{ f^i(x^i; x^{-i}) : x^i \in \mathcal{X}^i \}$$

$$\mathcal{X}^i := \{ g^i(x^i) \leq b^i, \;\; x^i \in \mathbb{Z}^{\alpha^i} \times \mathbb{R}^{\beta^i} \}$$

There is **common knowledge of rationality**, i.e., each player is **rational** and there is **complete information**

Köppe et al., 2011, Sagratella, 2015

Stable solutions: Nash equilibria



$\bar{x} = (\bar{x}^1, \ldots, \bar{x}^n)$ *is a Pure Nash Equilibrium (***PNE***) if, for any player* $\boldsymbol{i}$*,*

$$f^i(\bar{x}^i, \bar{x}^{-i}) \geq f^i(\hat{x}^i, \bar{x}^{-i}) \quad \forall \hat{x}^i \in \mathcal{X}^i$$

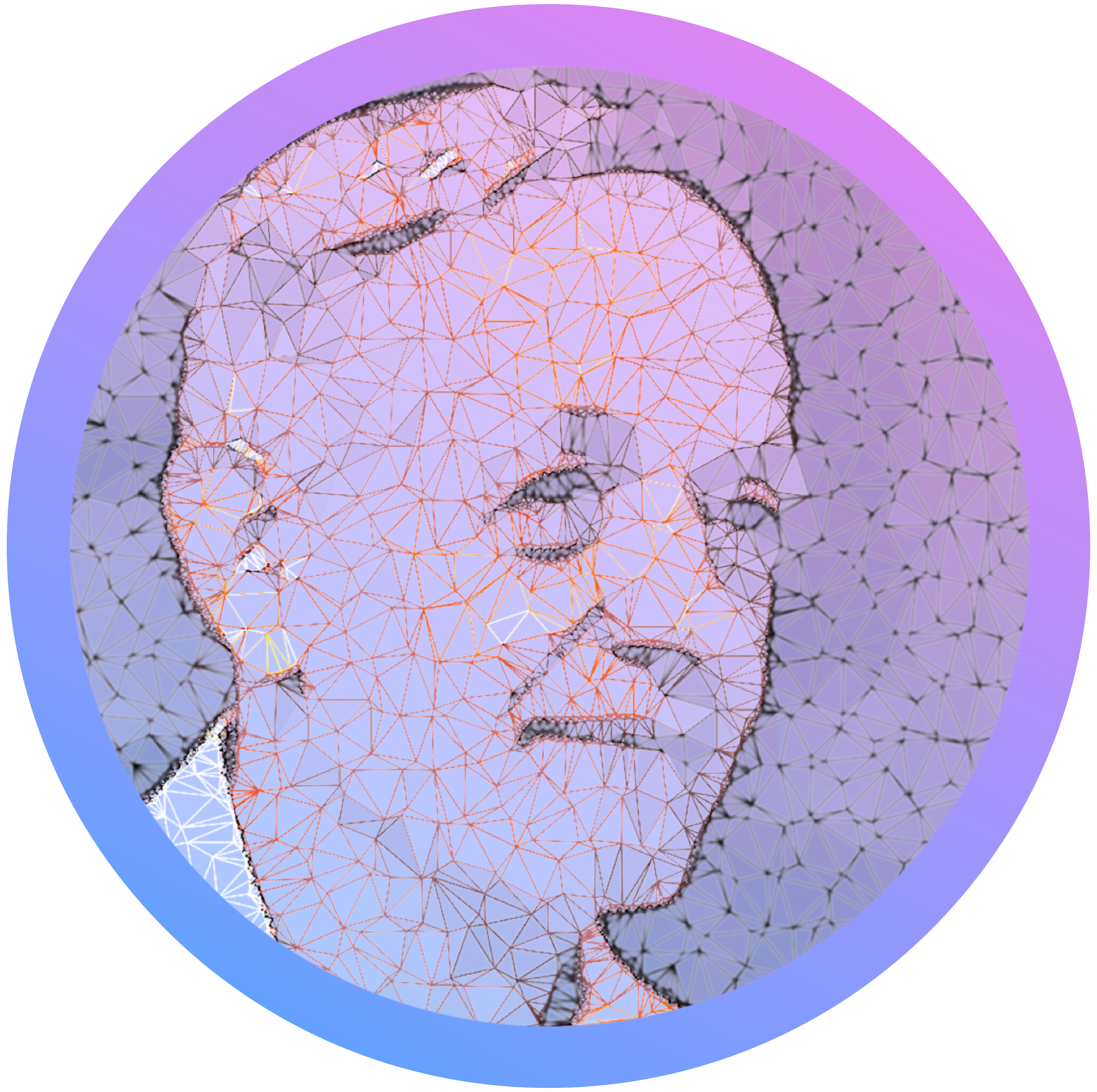
Mixed Nash equilibrium

Players **randomize** over their pure strategies
i.e., probability distribution over $\mathcal{X}^i$

Approximate equilibrium

Players can **deviate** up to an ϵ
e.g., relative or absolute deviation

Stable solutions: Nash equilibria



$\bar{x} = (\bar{x}^1, \ldots, \bar{x}^n)$ *is a Pure Nash Equilibrium* (**PNE**) *if, for any player* i,

$$f^i(\bar{x}^i, \bar{x}^{-i}) \geq f^i(\hat{x}^i, \bar{x}^{-i}) \quad \forall \hat{x}^i \in \mathcal{X}^i$$

PNEs and MNEs (**C.** *et al.*,

1. Deciding if an IPG has a pure equilibrium is Σ_2^p-complete
2. Deciding if an IPG has a mixed equilibrium is Σ_2^p-complete
3. If $\mathcal{X}^i$ is finite for any player i, there exists a mixed equilibrium

Why?

Decision-making is rarely an individual task

What if agents (players) decide by solving **mixed-integer optimization problems?**

- Modeling Capabilities
- Informative Solutions
- Practical Impact

Why? Modeling Capabilities

Modeling Capabilities

They extend traditional **resource-allocation tasks and combinatorial optimization** problems to a multi-agent setting

Indivisible quantities, **fixed-charge costs** and **logical implications** often require discrete variables

Energy — Gabriel et al., 2013, David Fuller and Çelebi, 2017
Supply Chain — Anderson et al., 2017
Assortment-Price competitions — Federgruen and Hu, 2015
Kidney Exchange Problems — **C.** et al., 2017
Cybersecurity — **D.** et al., 2023

Why? Informative Contents of Equilibria

Informative Solutions

Econometrica, Vol. 70, No. 4 (July, 2002), 1341–1378

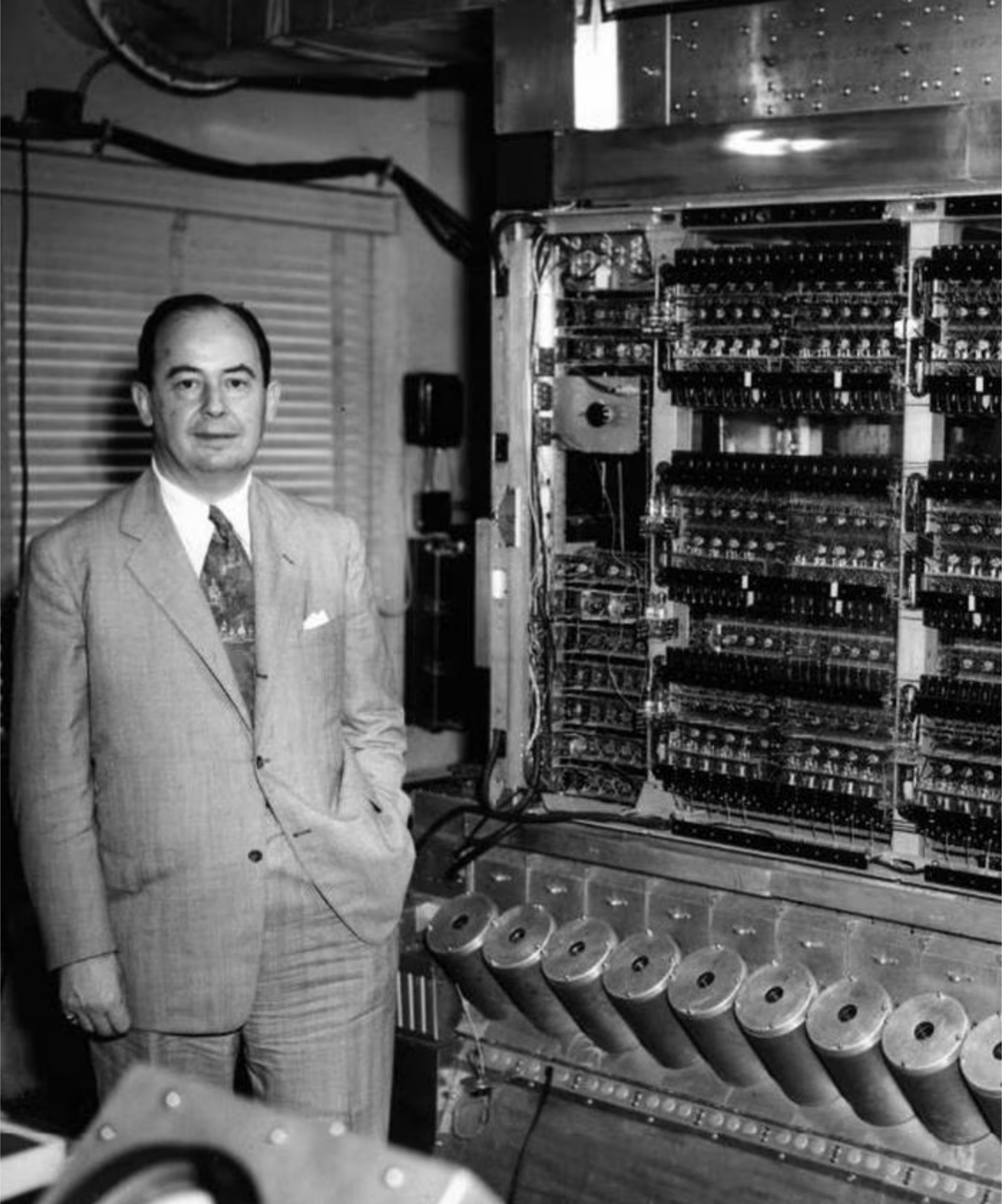
THE ECONOMIST AS ENGINEER: GAME THEORY, EXPERIMENTATION, AND COMPUTATION AS TOOLS FOR DESIGN ECONOMICS[1]

BY ALVIN E. ROTH[2]

*"Designers therefore **cannot work** only **with the simple conceptual models** used for theoretical insights into the general working of markets. Instead, **market design calls for an engineering approach**.*

***Experimental and computational economics** are natural complements to game theory in the work of design."*

How?



The bad news: non-convexity

$$\mathcal{X}^i := \{g^i(x^i) \leq b^i, \;\; x^i \in \mathbb{Z}^{\alpha^i} \times \mathbb{R}^{\beta^i}\}$$

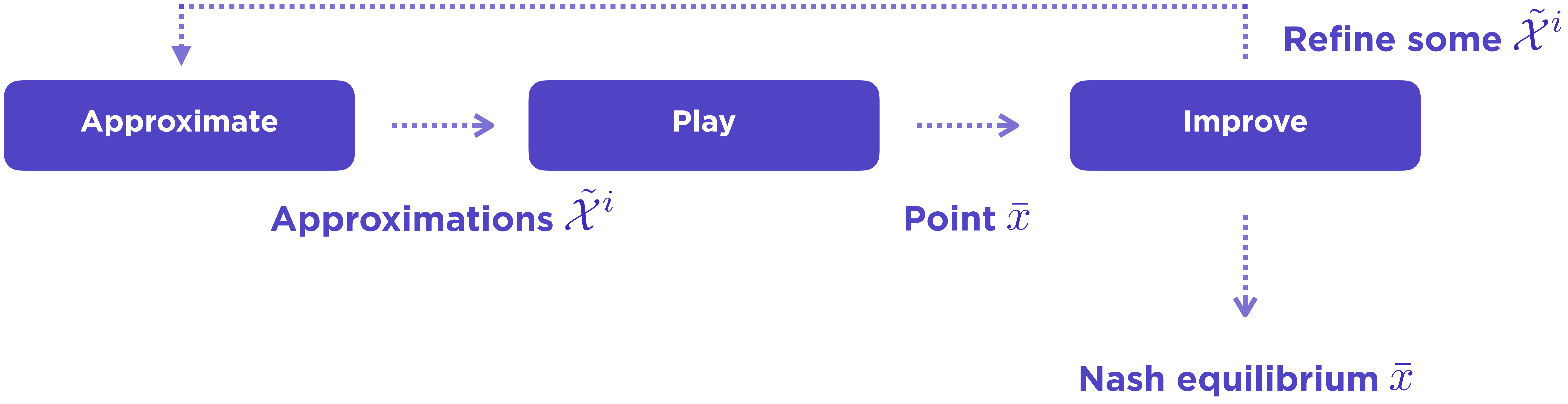
Historically, **convexity** played a central role in shedding light on the **existence and computation** of Nash equilibria

State-of-the-art

		Payoff Types f^i	Constraints $\mathcal{X}^i$
Sagratella, 2016	**Branching Method**	Convex payoffs	Bounded Convex Integer
C. et al., 2022	**Sample Generation Method**	Separable Payoffs	Bounded Mixed-Integer Linear
Schwarze and Stein, 2022	**Branch-and-Prune**	Quadratic Payoffs	Bounded Convex-Integer
C. et al., 2021	**Cut-And-Play**	Separable Payoffs	Polyhedral Convex-hull
Cronert and Minner, 2021	**Exhaustive Sample Generation Method**	Separable Payoffs	Bounded Pure-Integer
D. and Scatamacchia, 2023	**Zero Regrets**	Linearizable Payoffs	Bounded Mixed-Integer Linearizable

A 3-Phase Process

The **Approximate-Play-Improve** cycle



A 3-Phase Process

INPUT A game G described by the payoffs $f^i(x^i; x^{-i})$ and the actions $\mathcal{X}^i$

1 **Approximate**

Create $\tilde{G}$: Approximate $\mathcal{X}^i$ with some set $\tilde{\mathcal{X}}^i$

2 **Play**

Compute a solution $\bar{x}$ to $\tilde{G}$ by solving an **optimization problem**

3 **Improve**

Check if $\bar{x}$ is a Nash equilibrium

If not, refine at least a $\tilde{\mathcal{X}}^i$ and go to 2

Otherwise: **return the Nash equilibrium** $\bar{x}$

Phase 1: Approximate

1 **Approximate**

Create $\tilde{G}$: Approximate $\mathcal{X}^i$ with some set $\tilde{\mathcal{X}}^i$

The approximation $\tilde{\mathcal{X}}^i$ *should* exhibit **desirable properties** (e.g., convexity)

Computing the equilibria in $\tilde{G}$ is **relatively easier** compared to G

However, the approximation often **changes the structure of equilibria**

Outer Approximation

$$\tilde{\mathcal{X}}^i \supseteq \mathcal{X}^i$$

Inner Approximation

$$\tilde{\mathcal{X}}^i \subseteq \mathcal{X}^i$$

Phase 2: Play

2

Play

Compute a solution $\bar{x}$ to $\tilde{G}$ by solving an **optimization problem**

Compute a tentative equilibrium $\bar{x}$ for the approximated game $\tilde{G}$

Naturally depends on: the "***desirable properties***" of Phase 1

If $\tilde{\mathcal{X}}^i$ is convex, and f^i is concave in x^i —> **Complementarity Problem**

Phase 3: Improve

3 **Improve**

Check if $\bar{x}$ is a Nash equilibrium.

If not, refine at least a $\tilde{\mathcal{X}}^i$ and go to 2

Otherwise: **return the Nash equilibrium $\bar{x}$**

"*Separation theorem*" on steroids for Nash equilibria

Answer **two** questions:

Is $\bar{x}^i$ a **feasible strategy with respect to $\mathcal{X}^i$**?

Is $\bar{x}$ a **feasible Nash equilibrium**?

If one answer is negative, it provides a **"proof"**

Phase 3: Improve

3 **Improve**

Check if $\bar{x}$ is a Nash equilibrium

If not, refine at least a $\tilde{\mathcal{X}}^i$ and go to 2

Otherwise: **return the Nash equilibrium $\bar{x}$**

Is $\bar{x}$ a **feasible Nash equilibrium**?

Stability Step

No player should deviate from $\bar{x}^i$ given $\bar{x}^{-i}$

$$\tilde{x}^i = \arg\max_{x^i}\{f^i(x^i, \bar{x}^{-i}) : x^i \in \mathcal{X}^i\}$$

No **iff** $f^i(\bar{x}^i; \bar{x}^{-i}) < f^i(\tilde{x}^i; \bar{x}^{-i})$

Is $\bar{x}^i$ a **feasible strategy with respect to $\mathcal{X}^i$**?

Membership Step

If $\bar{x}^i \notin \mathcal{X}^i$, then refine $\tilde{\mathcal{X}}^i$

A 3-Phase Process

INPUT A game G described by the payoffs $f^i(x^i; x^{-i})$ and the actions $\mathcal{X}^i$

1 **Approximate**

Create $\tilde{G}$: Approximate $\mathcal{X}^i$ with some set $\tilde{\mathcal{X}}^i$

2 **Play**

Compute a solution $\bar{x}$ to $\tilde{G}$ by solving an **optimization problem**

3 **Improve**

Check if $\bar{x}$ is a Nash equilibrium

If not, refine at least a $\tilde{\mathcal{X}}^i$ and go to 2

Otherwise: **return the Nash equilibrium $\bar{x}$**

An example: Cut-and-Play

INPUT A game G described by the **separable** payoffs $f^i(x^i; x^{-i})$ and the actions $\mathcal{X}^i$

1 **Approximate**

Create $\tilde{G}$: Approximate $\mathcal{X}^i$ with some **polyhedron** $\tilde{\mathcal{X}}^i$

2 **Play**

Compute a solution $\bar{x}$ to $\tilde{G}$ via a **complementarity problem**

3 **Improve**

Check if $\bar{x}$ is a Nash equilibrium

If not, refine at an $\tilde{\mathcal{X}}^i$ via **branching/cutting** 2

Otherwise: **return the Nash equilibrium** $\bar{x}$

The Challenges

And some challenges

- Existence of equilibria
- How to select approximations
- Solving the play phase
- Equilibria Selection

And some challenges

Existence of equilibria

How to select approximations

Solving the play phase

Equilibria Selection

An equilibrium **might not exist** in a given game G.
How to detect infeasibility?

And some challenges

Existence of equilibria

How to select approximations

Solving the play phase

Equilibria Selection

An equilibrium **might not exist** in an approximation $\tilde{G}$ while one exists in the original game G (and vice versa)

Outer vs Inner approximations

And some challenges

- Existence of equilibria
- How to select approximations
- **Solving the play phase**
- Equilibria Selection

This phase often involves solving a **hard optimization problem**,
e.g., a mixed-integer program, variational inequality or complementarity problem

And some challenges

Existence of equilibria

How to select approximations

Solving the play phase

Equilibria Selection

If multiple equilibria exist, **selecting (i.e., optimizing) a specific one** is often complex from an algorithmic-design perspective

State-of-the-art

		Enumerate	Optimize	Payoff Types f^i	Constraints $\mathcal{X}^i$
Sagratella, 2016	Outer Approximation	✅	❌	Convex payoffs	Bounded Convex Integer
C. et al., 2022	Inner Approximation	❌	❌	Separable Payoffs	Bounded Mixed-Integer Linear
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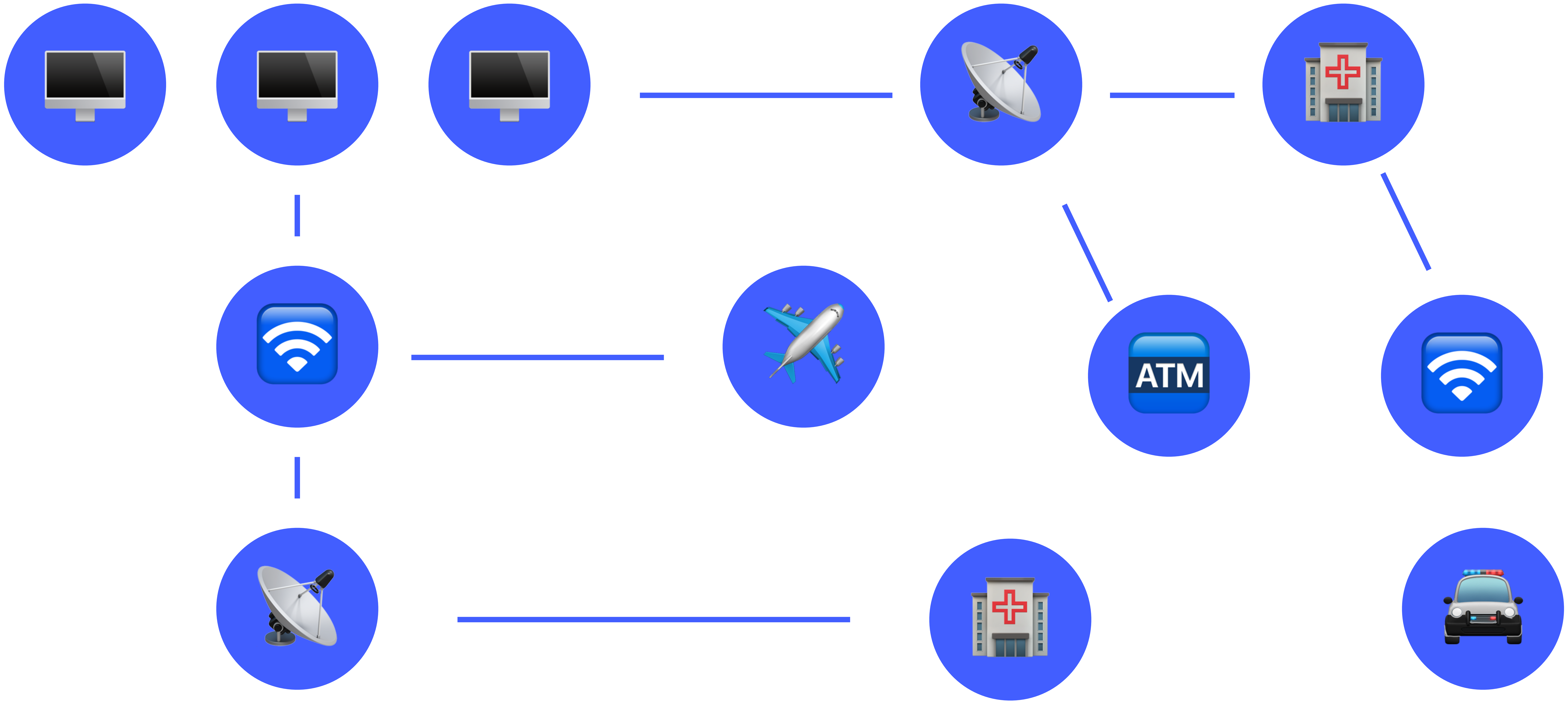
D. and Scatamacchia, 2023	Outer Approximation	✅	✅	Linearizable Payoffs	Bounded Mixed-Integer Linearizable

Protecting Critical Infrastructure

Critical Node Game



Critical Node Game

An attacker gain access to the network
at **an unknown node** (e.g., server)

Critical Node Game

An attacker gain access to the network at **an unknown node** (e.g., server)

The network operator decides **how to protect their network**

Critical Node Game

Critical Node Game

Players act simultaneously

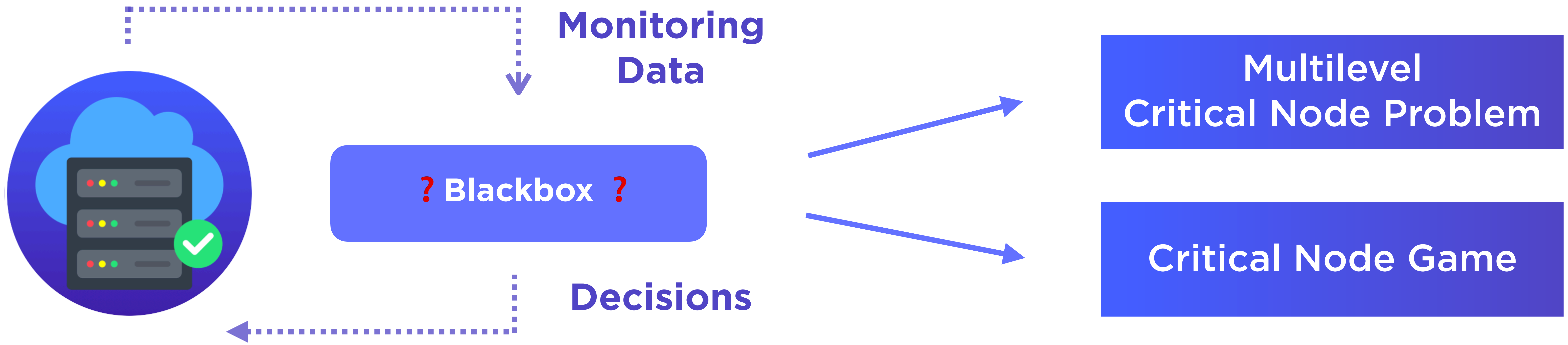
Multilevel Critical Node Problem

Players act sequentially

Real-time deployment of protective resources

A-priori **security posture assessment**

Critical Node Game



Mathematical Models

Attacker

The attacker has a **budget** A

Attacking $i \in R$ **costs** c_i^a and gives a **benefit** p_i^a

The variable α_i is 1 if the attacker attacks $i \in R$

Defender

The defender has a **budget** D

Defending $i \in R$ **costs** c_i^d and gives a **benefit** p_i^d

The variable x_i is 1 if the defender protects $i \in R$

Mathematical Models

x_i	α_i	Payoff^D	Payoff^A
0	0	p_i^d	$-\gamma p_i^a$
0	1	δp_i^d	p_i^a
1	0	ϵp_i^d	0
1	1	ηp_i^d	$(1-\eta)p_i^a$

$\gamma \in [0,1]$: **opportunity cost** of not attacking

$\delta \in [0,1]$: degradation with **ongoing attack**

$\varepsilon \in [0,1]$: degradation with **passive mitigation**

$\eta \in [0,1]$: degradation with **active mitigation**

Mathematical Models

Critical Node Game

Simultaneous IPG

$$\max_{x} \left\{ f^d(x;\alpha) : d^\top x \leq D, x \in \{0,1\}^{|R|} \right\}$$

$$\max_{\alpha} \left\{ f^a(\alpha;x) : a^\top \alpha \leq A, \alpha \in \{0,1\}^{|R|} \right\}$$

Multilevel Critical Node Problem

Bilevel Game

$$\begin{aligned} \max_{x,\hat{\alpha}} \quad & f^d(x,\hat{\alpha}) \\ \text{s.t} \quad & d^\top x \leq D, \\ & x \in \{0,1\}^{|R|}, \\ & \hat{\alpha} \in \arg\max_{\alpha} \left\{ f^a(\alpha;x) : a^\top \alpha \leq A, \alpha \in \{0,1\}^{|R|} \right\}. \end{aligned}$$

Metrics

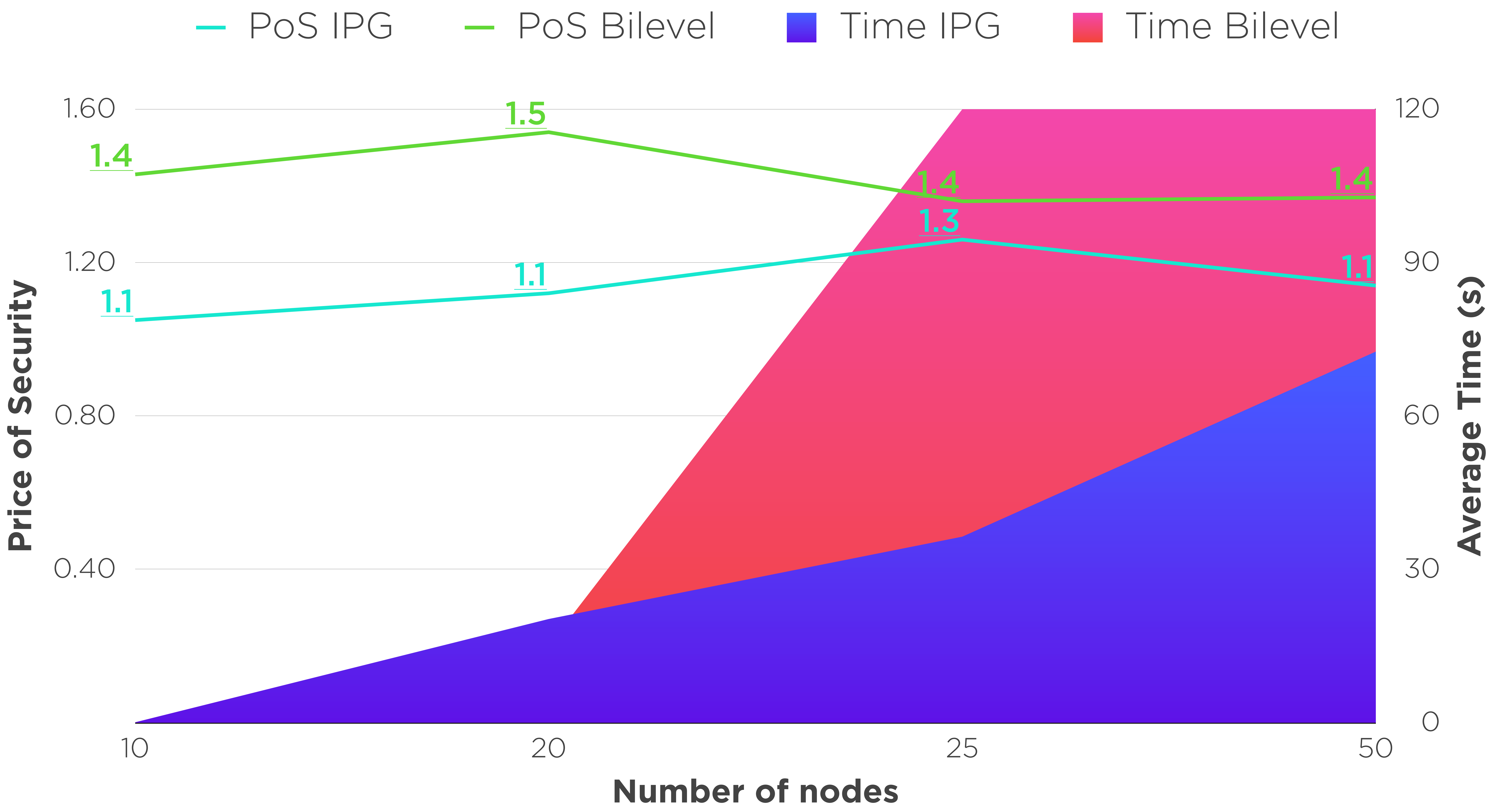
We measure time, and the **effectiveness of the defender's strategy**

$$\textbf{Price of Security}\ (\bar{x}, \bar{\alpha}) = \frac{\text{Best defender's objective for any outcome}}{\textbf{Defender's objective in } (\bar{x}, \bar{\alpha})}$$

All the tests and algorithms of this tutorial are implemented in our open-source package **ZERO**

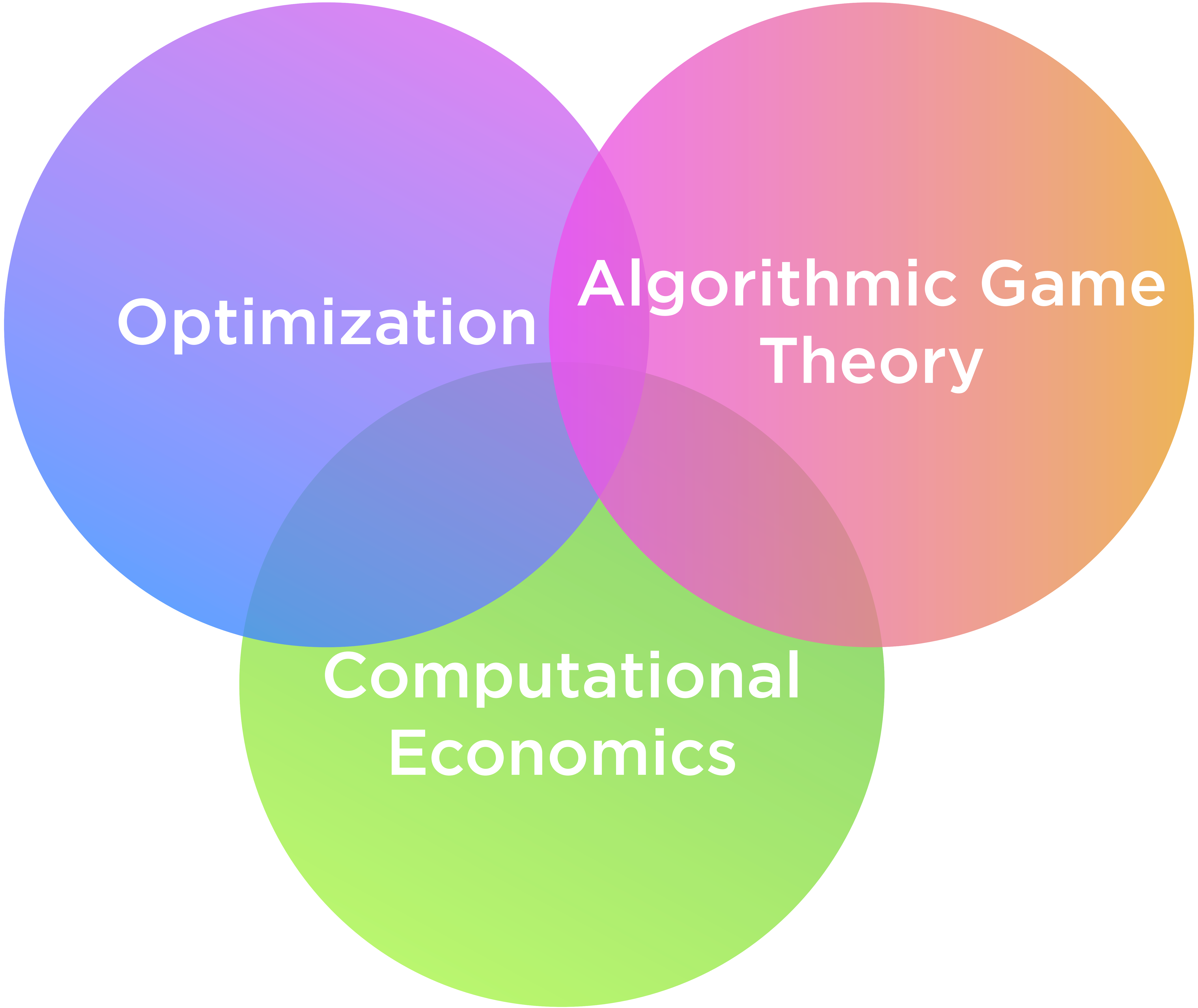
https://github.com/ds4dm/ZERO

Results



PoS IPG
PoS Bilevel
Time IPG
Time Bilevel
1.60
1.20
0.80
0.40
120
90
60
30
0
1.4
1.5
1.4
1.4
1.3
1.1
1.1
1.1
1.1
Price of Security
Average Time (s)
10
20
25
50
Number of nodes

Looking Ahead


Optimization
Algorithmic Game Theory
Computational Economics

Decision-making is rarely an individual task

Toolbox to model competitive decision-making with **mixed-integer optimization**

- **Modeling Capabilities**
- **Informative Solutions**
- **Practical Impact**

There are still (too) many open questions

Some perspectives

Applications

Deployment of IPGs in **new application domains**

Optimization

Develop **theoretically-grounded and efficient** both general and problem-specific algorithms

Fairness

Solutions **balancing** the decision-makers selfishness with **societal goals**

arXiv 2303.11188

Integer Programming Games:
A Gentle Computational Overview

INFORMS TutORial in Operations Research, 2023

Margarida

Gabriele

Andrea

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